Problem 1. Let \((X, \rho)\) be a metric space. Let \(E \subset X\).

1.1 Prove that every uniformly convergent sequence of bounded functions on \(E\) is uniformly bounded on \(E\).

1.2 Assume that \(\{f_n\}\) and \(\{g_n\}\) are sequences of bounded functions which converge uniformly on \(E\). Prove that \(\{f_ng_n\}\) converges uniformly on \(E\).

1.3 Assume that \(\{f_n\}\) and \(\{g_n\}\) converge uniformly on \(E\). Does \(\{f_ng_n\}\) converge uniformly on \(E\)?

Problem 2. Let \((X, \rho)\) be a metric space. Let \(K \subset X\) be compact. For each \(k \in \mathbb{N}\), let \(f_k : K \to \mathbb{R}\) be continuous and satisfy:

(i) \(f_k(x) \geq 0\) for \(x \in K\).

(ii) \(f_k(x) \to 0\) pointwise in \(K\).

(iii) \(f_k(x) \leq f_\ell(x)\) for \(x \in K\) whenever \(k \geq \ell\).

Prove that \(f_k(x) \to 0\) uniformly on \(K\).

Problem 3 Let \((X, \rho)\) be a non-empty complete metric space. Let \(f : X \to X\) be such that there exists \(\lambda \in (0, 1)\) satisfying

\[
\rho(f(x), f(y)) \leq \lambda \rho(x, y) \quad \text{for all } x, y \in X.
\]

Prove that there exists a unique point \(u \in X\) such that \(f(u) = u\). (Hint: let \(x \in X\), and consider the sequence \(x, f(x), f(f(x)), \ldots\). This result is known as Banach’s fixed-point theorem. It implies several existence theorems in the theory of partial differential equations.

Problems from Royden:

Chapter 7, Section 8: problem 43.
Chapter 7, Section 10: problems 47, 50.
(*) Problem. Let \((X, \rho)\) be a metric space, and let
\[
C_1(X) = \{ f : X \to \mathbb{R}, \ f \text{ continuous} : \sup_{x \in X} |f(x) - \rho(x, x_0)| < \infty \},
\]
for some chosen \(x_0 \in X\). For \(f, g \in C_1(X)\), define
\[
\sigma(f, g) = \sup_{x \in X} |f(x) - g(x)|.
\]

1. Show that \((C_1(X), \sigma)\) is a complete metric space which is independent of the choice of \(x_0\).

2. Define the map \(\iota : X \to C_1(X)\) by \(\iota(x) = \rho_x\), where \(\rho_x\) denotes the function \(\rho_x(y) = \rho(x, y)\). Show that \(\iota\) is an isometric embedding of \(X\) into \(C_1(X)\).