Problem 1. (Gagliardo-Nirenberg-Sobolev inequality) Let $1 \leq p < n$, and let $B \subset \mathbb{R}^n$ denote the ball of center 0 and radius 1. Show that there exists a constant $C$ depending only on $n$ and $p$ such that
\[ \|u\|_{\frac{np}{n-p}} \leq C\|Du\|_p, \]
for all $u \in C^1_c(B)$. Here
\[ \|f\|_q = \left(\int_B |f|^q\right)^{\frac{1}{q}}. \]
Hint: Recall the fundamental theorem of calculus and consider the case $p = 1$ first.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $p \geq 2$.
2.1. Show that for $f, g \in L^p(\mu)$,
\[ \left\| \frac{f + g}{2} \right\|_p^2 + \left\| \frac{f - g}{2} \right\|_p^2 \leq \frac{1}{2} \|f\|_p^2 + \frac{1}{2} \|g\|_p^2. \]

2.2. Let $\{f_n\} \subset L^p(\mu)$ and $f \in L^p(\mu)$. Suppose that $\|f_n\|_p = \|f\|_p = 1$ and that
\[ \lim_{n \to \infty} \left\| \frac{f_n + f}{2} \right\|_p = 1. \]
Show that
\[ \lim_{n \to \infty} \|f_n - f\|_p = 0, \]
i.e. $L^p(\mu)$ is locally uniformly convex.

Problem 3. (Radon-Riesz Theorem) Let $\{f_n\} \subset L^p(\mu)$ and $f \in L^p(\mu)$. Let $p \geq 2$. Suppose that $f_n \rightharpoonup f$ in $L^p(\mu)$ and that $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Prove that
\[ \lim_{n \to \infty} \|f_n - f\|_p = 0. \]
Hint: Use problem 2.
Problem 4. Let $n \geq 3$. Let $B \subset \mathbb{R}^n$ denote the ball of center 0 and radius 1. Let $P = \varphi X$ where $\varphi \in C^1(B)$ and $X \in C^1(B, \mathbb{R}^n)$.

4.1. Show that
$$\text{div } P = \varphi \text{ div } X + \langle \nabla \varphi, X \rangle.$$  

4.2. Let $u$ be a harmonic function in $B$, i.e. $\Delta u = 0$ in $B$. Denote by $B_r$ the ball of center 0 and radius $r$. Show that for $r \in (0, 1)$

$$(\ast) \quad r \int_{\partial B_r} |\nabla u|^2 = 2r \int_{\partial B_r} \langle \nabla u, \nu \rangle^2 + (n - 2) \int_{B_r} |\nabla u|^2,
$$

where $\nu = \frac{X}{|X|}$, and $X$ denotes the position vector.

Hint: Consider
$$\text{div } (X|\nabla u|^2) - 2\text{div } (\langle X, \nabla u \rangle \nabla u).$$

Remark: The monotonicity formula above ($\ast$) is an useful tool when studying the order of vanishing of harmonic functions.