Problem 1. Let $(X, d)$ be a compact metric space. We denote by $\mathcal{C}(X)$ the family of closed subsets of $X$. For $A, B \in \mathcal{C}(X)$ define the Hausdorff distance between $A$ and $B$ by,

$$D[A, B] = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ denotes the distance from the point $a$ to the set $B$, and $d(b, A) = \inf\{d(b, a) : a \in A\}$ the distance of the point $b$ to the set $A$.

1.1 Show that $D$ defines a metric on $\mathcal{C}(X)$.

1.2 Prove that $(\mathcal{C}(X), D)$ is a complete metric space.

1.3 Prove that $(\mathcal{C}(X), D)$ is totally bounded; that is, for every $\epsilon > 0$ there exists a finite cover of $\mathcal{C}(X)$ by $\epsilon$-balls for the metric $D$.

Together, these imply that $(\mathcal{C}(X), D)$ is a compact metric space.

Let $(X, \rho)$ be a metric space. Let $E \subset X$.

• A point $x \in X$ is a limit point or an accumulation point of $E$ if $\forall r > 0$, $E \cap B(x, r) \setminus \{x\} \neq \emptyset$.

• $E$ is said to be perfect if $E$ is closed and if every point of $E$ is an accumulation point of $E$.

Problem 2. Prove that a nonempty perfect set in $\mathbb{R}^n$ is uncountable.

Problems from Royden: Chapter 7, Section 4: problems 14, 15.

Chapter 7, Section 5: problem 21.

Chapter 7, Section 7: problem 27.