1. Find integers $x, y \in \mathbb{Z}$ so that $12x + 29y = 1$.

$\begin{array}{l} a \qquad\quad b \\ 29 = 2 \cdot 12 + 5 \\ 12 = 2 \cdot 5 + 2 \\ 5 = 2 \cdot 2 + 1 \end{array}$

$$a = 2b + (a - 2b)$$

$$b = 2(a-2b) + \underbrace{b - 2(a-2b)}_{5b - 2a}$$

$$a - 2b = 2(5b - 2a) + \underbrace{a - 2b - 2(5b - 2a)}_{\underset{29 \quad\;\; 12}{5a - 12b}}$$

$$\boxed{\begin{array}{l} x = -12 \\ y = 5 \end{array}}$$

2. Only one of the following two statements is true:

False **Statement A:** "The sum of any 4 consecutive integers is divisible by 4." $1+2+3+4=10$, $4 \nmid 10$

true **Statement B:** "The sum of any 5 consecutive integers is divisible by 5."

Identify which statement is true and which statement is false. Then, provide a proof for the true statement.

Proof of statement B: Let $n, n+1, n+2, n+3, n+4$ be 5 consecutive integers. Then

$$n+(n+1)+(n+2)+(n+3)+(n+4) = 5n+10$$
$$= 5(n+2),$$

which is divisible by 5.

3. Find the smallest number with exactly 20 divisors.

$$20 = \tau(n) = (e_1+1)\cdots(e_k+1)$$

where $n = p_1^{e_1} \cdots p_k^{e_k}$

Try $20 = 5 \cdot 2 \cdot 2$, so $e_1 = 4$, $e_2 = 1$, $e_3 = 1$

The smallest n is achieved with $p_1 = 2, p_2 = 3, p_3 = 5$.

$$n = 2^4 \cdot 3^1 \cdot 5^1 = \boxed{240}$$

Also try $20 = 5 \cdot 4$ with $e_1 = 4, e_2 = 3$, but $n = 2^4 \cdot 3^3$ is larger.

4. In this problem a, b, and n are positive integers.

(a) If $a|n$, $b|n$, and $\gcd(a,b) = 1$, prove that $ab|n$.

$\gcd(a,b) = 1 \Rightarrow ax + by = 1$ for some $x, y \in \mathbb{Z}$.
Multiply by n: $anx + bny = n$. By assumption, both terms on the left are divisible by ab, so the same is true for n.

(b) Give an example to show that the statement in part (a) is not necessarily true if you remove the "$\gcd(a,b) = 1$" condition.

Take $a = b = n = 2$.

5. Choose **one** of the following statements to prove (both are true!)

   (a) For any $k > 1$, there are infinitely many n such that $\tau(n) = k$.

   (b) $\tau(n) \leq 2\sqrt{n}$ for all $n \geq 1$.

(a) The number p^{k-1} satisfies $\tau(p^{k-1}) = k$, for p prime. The desired result then follows from the infinitude of primes.

(b) Any divisor $d \mid n$ with $d \geq \sqrt{n}$ can be paired with the divisor $\frac{n}{d}$ which is necessarily $\leq \sqrt{n}$. But there are at most $\sqrt{n}$ divisors $\leq \sqrt{n}$, so there are at most $2\sqrt{n}$ divisors in total.

6. (★) A number n is called abundant if $\sigma(n) > 2n$. Let's call a number n **hyperabundant** if $\sigma(n) > 100n$.

Do hyperabundant numbers exist? Either prove or disprove the existence of such numbers.

Let $n = k!$ (these numbers have lots of divisors)

Summing all the divisors of n in order, we have:

$$\sigma(n) = 1 + 2 + 3 + \cdots + k-1 + k + (\text{other divisors}) + \underbrace{\frac{n}{k} + \frac{n}{k-1} + \cdots + \frac{n}{3} + \frac{n}{2} + \frac{n}{1}}_{\text{just take these}}.$$

(Note: for $k > 3$, $k < \frac{k!}{k}$ so all the listed divisors are distinct.)

Then $\sigma(n) > \frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{k-1} + \frac{n}{k} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)$

Since the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, choosing k sufficiently large will make $n = k!$ hyperabundant.