MATH 301: Problem Set 8

1. For practice, compute the following values of Euler's phi function: $\varphi(21)$, $\varphi(42)$, $\varphi(43)$, and $\varphi(7!)$.

2. Prove that $\varphi(n) = \varphi(2n)$ if and only if n is odd.

3. Show that the product formula for $\varphi(n)$ can be rearranged to obtain the expression
$$\frac{\varphi(n)}{n} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$
Then, interpret this as a statement about *probabilities.*

4. In lecture 13, we proved that a prime dividing a number of the form n^2+1 is necessarily of the form $4k+1$ (well, aside from the prime 2).

   (a) Suppose instead of n^2+1 we consider n^2-1. As it turns out, the question of which primes can appear as divisors of n^2-1 is much less interesting. Why?

   (b) Suppose we consider n^2-5 instead. Write down the numbers n^2-5 for $n = 3, 4, \ldots, 12$ and then find the prime factorization for each.

   (c) Based on your data, make a **conjecture** about which primes can appear as divisors of n^2-5.

5. Find all values of n such that $\varphi(n) = 32$. There are many!

6. A number n is called **quasiperfect** if $\sigma(n) = 2n+1$. Here $\sigma(n)$ is the sum of divisors function.

   (a) (★) Prove that a quasiperfect number must be an odd square.

   (b) (★★) Do any quasiperfect numbers exist?