MATH 301: Problem Set 4

1. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ has the form } 4k+1 \\ -1 & \text{if } n \text{ has the form } 4k+3 \end{cases}$$

   Prove that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$ (that is, f is *completely multiplicative*)

2. Find the smallest number with exactly 60 divisors.

3. A hallway at a school contains 100 lockers numbered 1 to 100. The first student goes along and opens every locker. The second student then closes every second locker, starting with locker number 2. The third student changes the state of every third locker, opening it if it's closed and closing it if it's open. The fourth student changes the state of every fourth locker, and so on until all 100 students have followed the same pattern.

   Which lockers remain open at the end?

4. Find all numbers n such that $\sigma(n) = 248$.

5. A number n is called **abundant** if $\sigma(n) > 2n$.

   (a) Prove that any multiple of an abundant number is abundant.

   (b) The first 3 even abundant numbers are 12, 18, and 20. On the other hand, there is only one odd abundant number less than 1000. Find it! (Hint: try to create a prime factorization of an odd number that has as many divisors as possible while also being as small as possible)

6. (★) The square numbers 529 and 729 differ by exactly 200. Are there any other pairs of square numbers that differ by exactly 200? If so, find all such pairs. (Hint: this problem is basically asking you to solve a certain Diophantine equation.)