MATH 301: Problem Set 3

1. Consider the number system $\mathbb{E}$ consisting of all positive even integers. An element $p \in \mathbb{E}$ is called "prime" if it *cannot* be written as a product $p = ab$ where both a and b are in $\mathbb{E}$.

   Find the smallest element of $\mathbb{E}$ that can be written as a product of "primes" in 2 different ways. (This shows that unique factorization fails for the number system $\mathbb{E}$!)

2. Find the smallest positive integer n such that $15120n$ is a perfect square. [Hint: how would you identify a number as square from its prime factorization?]

3. Prove that there are infinitely many primes of the form $6n + 5$.

4. The number 100! ends with a long string of zeroes. How many zeroes exactly?

5. Prove there exist 1 million consecutive composite numbers.

6. This problem concerns the mysterious polynomial $f(x) = x^2 + x + 41$.

   (a) Compute $f(n)$ for $n = 1, 2, \ldots 10$. Based on the type of numbers you're getting, make a conjecture about the possible values $f(n)$ when n is any positive integer.

   (b) Either prove or disprove your conjecture from part (a).

   (c) (★) In fact, prove that for any polynomial of the form $f(x) = ax^2 + bx + c$ with a, b, c integers and $a \neq 0$, $f(n)$ will be *composite* for infinitely many positive integers n.

   (d) (★★) Find a quadratic polynomial $f(x)$ such that $f(n)$ is prime for infinitely many values of n. This is still a double starred problem, even though you're allowed to choose the polynomial!