MATH 301: Problem Set 1

1. Compute the sums 1, $1 + 3$, $1 + 3 + 5$, and $1 + 3 + 5 + 7$. *Guess* a formula for the sum of the first n odd numbers.

   Now prove your formula in 2 different ways: a) with induction, and b) with a picture. Bonus points if your picture contains no words or mathematical symbols!

2. Prove the identity
$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1},$$
   where F_n indicates the nth Fibonacci number.

3. What is the largest postage value that *cannot* be achieved using 5¢ and 8¢ stamps? Can you prove your answer?

4. *Aliquot chains.* Starting from any positive integer we can form a chain by repeatedly following the rule: to get the next number in the chain, take the sum of proper divisors of the previous number.

   As an example let's compute the aliquot chain for 10. The proper divisors of 10 are 1, 2, and 5 (we don't include 10 itself), so the next element of the chain is $1 + 2 + 5 = 8$. The proper divisors of 8 are 1, 2, and 4, so the next element of the chain is $1 + 2 + 4 = 7$. As 7 is prime, the next and final element of the chain is the sum of its only proper divisor, 1. So the chain for 10 looks like this:
$$10 \to 8 \to 7 \to 1.$$
   Now you try it! Compute aliquot chains for the following numbers:

   a) 12
   b) 6
   c) 220
   d) 30

5. (★★) What is the eventual fate of the aliquot chain for 276?

6. (★) Prove that any positive integer can be expressed as a sum of distinct, *nonconsecutive* Fibonacci numbers. By nonconsecutive I mean not occurring next to each other in the sequence $1, 2, 3, 5, 8, 13, 21$, etc.

   (so $1 + 3 + 13$ is a valid expression for 17, but $1 + 3 + 5 + 8$ and $2 + 2 + 13$ are not)