0/1 Polytopes Survey

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1 An Introduction to 0/1 Polytopes

Though the notion of a 0/1 polytope is far from difficult to define, we will see that these seemingly innocent objects can be extremely complex. In particular, low-dimensional examples tend to obscure the truth. We begin with the basic definitions of the subject.

Definition 1.1. A **0/1 polytope** is the convex hull of a subset of $\{0,1\}^d$, the vertices of the d-cube. More precisely, it is a set $P \subseteq \mathbb{R}^d$ such that $P = P(V) = \{V\mathbf{x} | x \geq 0, \mathbf{1}^T\mathbf{x} = 1\}$, where V is a 0/1-matrix such that the columns are a subset of the vertex set of $[0,1]^d$, the unit d-cube.

This particular presentation in terms of vertices is commonly called a $\mathcal{V}-$presentation. As with all polytopes, there is an equivalent inequality representation, which we term an $\mathcal{H}-$presentation. As noted above, these objects are quite easy to define, yet their properties tend to be quite unwieldy in higher dimensions.

1.1 Equivalences

Definition 1.2. There are several types of equivalences for 0/1 polytopes:

1. Two 0/1 polytopes in $\mathbb{R}^d$ are said to be **0/1 equivalent** if one can be achieved via a finite number of rotations and inversions on the d-cube.
2. The set of all faces of a polytope P, partially ordered by inclusion, is called the **face lattice** of P. We say that two polytopes are **combinatorially equivalent** if they have isomorphic face lattices.
3. Two polytopes $P_1 \subset \mathbb{R}^d$ and $P_2 \subset \mathbb{R}^e$ are **affinely equivalent** (or *affinely isomorphic*) if there exists an affine map $f : \mathbb{R}^d \to \mathbb{R}^e$ such that f maps the points of P_1 bijectively onto the points of P_2. We say that f is an *affine map* if it has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$ for some $\mathbf{t} \in \mathbb{R}^e$ and some matrix $A \in \mathbb{R}^{e \times d}$.

The following result is surprising not in its implications, but in the fact that *all* of the reverse implications are false.

Proposition 1.3. *For all* $0/1-$*polytopes in* $\mathbb{R}^d$*, the following implications hold:*

0/1 Equivalent $\Rightarrow$ ***Congruent*** $\Rightarrow$ ***Affinely Equivalent*** $\Rightarrow$ ***Combinatorially Equivalent***

with the reverse implications false in each case.

We will show only a select counterexample for the reverse implications, so for the other counterexamples and a proof of the forward implications, see [Zie97]. Consider the following example (in polymake vertex input format) of two congruent full-dimensional 0/1-polytopes in dimension 5:

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

This can be checked to be congruent by calculating the pairwise distance of the points. Note, however, that if these were 0/1-equivalent, then we could arrive at one from the other by a series of permutating and complementing the columns, but this is clearly not possible in the example.

A natural question is how many combinatorial equivalence classes of 0/1-polytopes there are in dimension d. Clearly there is an upper bound of 2^{2^d}, since this is the precise number of 0/1-polytopes in $\mathbb{R}^d$. However, some or many of these may be equivalent. The answer, however, is not far from this upper bound. As stated in [Zie97, Proposition 8], there is a doubly-exponential lower bound of combinatorially inequivalent 0/1-polytopes in $\mathbb{R}^d$. Specifically, for $d \geq 6$, there are more than $2^{2^{d-2}}$ combinatorially non-equivalent 0/1-polytopes.

2 Combinatorics

Though $\mathcal{V}-$presentation of a 0/1-polytope is fairly compact, many methods of combinatorial optimization involve computing, in part or in whole, the $\mathcal{H}-$presentation of the polytope. That is a matrix A and vector $\mathbf{b}$ such that $P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$, which is in general larger than the size of the $\mathcal{V}-$presentation. So the motivating question is, "How bad could it be?" There are at least two contributing factors:

1. How many facets can a 0/1-polytope of dimension d have, that is the minimum number of rows of A?

2. How large can the size of an entry in A be?

The answer to (1) follows, and (2) will be discussed in §3.

2.1 Facet numbers

Let $\#f(d)$ be maximal number of facets of a d-dimensional 0/1 polytope. Our recurring theme of "low-dimensional intuition does not work" appears again here. Though it may seem in studying the case up to and including $d = 4$ that the answer is precisely 2^d, the $d = 5$ case was shown by Ewgenij Gawrilow to be at least $\#f(5) \geq 40$. A result of Aichholzer showed this to be an equality, but the pattern of 2^d clearly ends here. Hence, as Ziegler puts it, 0/1-polytopes may have *many* facets.

The best known upper bound is actually factorial in n, and is thought to be fairly tight:

Theorem 2.1 (Fleiner, Kabiel and Route)**.** *For all large enough d, a d-dimensional 0/1-poltyope has no more than*

$$\#f(d) \leq 30(d-2)!$$

facets.

At the time of Ziegler's paper, the best known lower bound for $\#f(d)$ was (Kortenkamp et al): $\#f(d) > 3.6^d$. Ziegler believed that this result was not very sharp at all, and in his paper suggested two lines of attack for proving a better lower bound: (1) random 0/1-polytopes, and (2) cut polytopes. It turned out that the latter approach was more useful, as a few years later, Bárány and Pór [BP01] used random 0/1-polytope theory to prove a much better lower bound.

Theorem 2.2 (Bárány and Pór)**.** *For $\#f(d)$ as defined above, $\#f(d) \geq (cd/\log d)^{(d/4)}$ for some constant c.*

The proof of this theorem will be addressed in §2.2. Zong [Zon05] in his survey summarizes these upper and lower bounds as $\log(\#f(d)) = \Theta(d \log d)$. In other words, there are constants c_1 and c_2 such that

$$c_1 d \log d \leq \log(\#f(d)) \leq c_2 d \log d.$$

When viewed in this way, it appears that these bounds are quite nice.

2.2 Random 0/1 polytopes

Bárány and Pór [BP01] use the theory of random 0/1-polytopes in their proof of their result. Ziegler [Zie97, §3] discusses some of the known results regarding random 0/1-polytopes. As in most areas of extremal combinatorics, there are several models for random polytopes. The model discussed by Ziegler, and use by Bárány and Pór, is the following: Fixing a dimension d, choose n 0/1-points in $\mathbb{R}^d$ independently with uniform probablility. That is, for each of the n choices, each of the 2^d possible 0/1-points in $\mathbb{R}^d$ has probability 2^{-d} of being chosen. (Note that for $n > 1$ it is possible that a given point may be chosen more than once, i.e. repeated.) The random polytope is then the convex hull of the chosen points.

The fundamental fact for analyzing this model is the parameter P_d, the probability that a random $d \times d$ 0/1-matrix has determinant equal to zero. By a theorem of Williamson [Zie97, Prop. 11], this turns out to be equal to the probability that a random 0/1-polytope obtained by choosing $d+1$ points in $\mathbb{R}^d$ is *not* a full d-dimensional simplex. Computations and estimates for $d \leq 15$ have shown that $P_1 = 0.5 < P_2 < P_3 = 0.66015625$, and that thereafter P_d decreases steadily, apparently approaching zero asymptotically. It may be "intuitively obvious" that P_d should approach zero as a limit, but the proof is surprisingly non-trivial. This convergence was first proved by Komlós. The best known bounds on P_d are the following.

Theorem 2.3 (Komlós's Theorem; Kahn, Komlós, and Szemerédi)**.** *The probability P_d that a random $d \times d$ 0/1-matrix is singular, for all sufficiently large d, satisfies*

$$\frac{d^2}{2^d} < P_d < 0.999^d$$

The non-trivial bound is the upper one. It has been conjectured that the lower bound is closer to the truth, which can be interpreted as saying that if a random $d \times d$ 0/1-matrix is singular, then most likely there are two columns or two rows that are equal.

Some properties of random 0/1-polytopes follow from Komlós's Theorem [Zie97, Cor. 14]:

Corollary 2.4. *With high probability, i.e. probability that tends to 1 as $d \to \infty$, the following are true:*

(i) Any polynomial (in d) number of 0/1-vectors chosen randomly (independently and with equal probability) from $\{0,1\}^d$ will be distinct.

(ii) The affine hull of a set of d randomly chosen 0/1-points in $\mathbb{R}^d$ is a hyperplane that does not contain the origin $\mathbf{0}$.

(iii) The convex hull of $d+1$ randomly chosen 0/1-points in $\mathbb{R}^d$ is a d-dimensional simplex.

This suggests other questions, such as *What is the expected volume of a random simplex?* As Ziegler summarizes, this tends to be huge, compared to d. An exercise in the paper is to show that for a random $d \times d$ 0/1-matrix A, the expected value of the square of the determinant of A is exactly

$$E(\det(A)^2) = \frac{(d+1)!}{2^{2d}}.$$

But if that is the *expected* value for the square of the determinant, then it is quite common (to be expected) that for many such random matrices A,

$$|\det(A)| \geq \frac{\sqrt{(d+1)!}}{2^d}$$

.

Theorem 2.5 (Füredi (Prop. 15 in Ziegler))**.** *For any constant $\epsilon > 0$, with high probability (i.e. tending to 1 as $d \to \infty$), a random 0/1-polytope in $\mathbb{R}^d$ with $n \geq (2+\epsilon)d$ vertices contains $\frac{1}{2}\mathbf{1}$, while a random 0/1-polytope in $\mathbb{R}^d$ with $n \leq (2-\epsilon)d$ vertices does not contain $\frac{1}{2}\mathbf{1}$.*

This theorem is related to another set of results by Dyer, Füredi, and McDiarmid which play a key role in Bárány and Pór's proof. The idea, loosely stated, is the following: for large enough d, and for n in an appropriate range, consider the function that assigns to each point $\mathbf{x}$ in the unit hypercube the probability that $\mathbf{x}$ will lie in the 0/1-polytope in $\mathbb{R}^d$ formed from n randomly chosen vertices. This probability function is not uniform. It is greatest at $\frac{1}{2}\mathbf{1}$, and smaller near the vertices of the hypercube. It is also a convex function, and has a sharp "knee" at a particular distance (under a particular metric) from $\frac{1}{2}\mathbf{1}$ which is dependent on n. Bárány and Pór exploit this to find two closely nested convex bodies $B_1 \subset B_2$ in the hypercube such that with high probability a random 0/1-polytope K_n with n vertices contains B_1 but does not contain a certain proportion r of the boundary ∂B_2 as measured by $(d-1)$-dimensional volume. If H is a halfspace that contains B_1, the proximity of ∂B_1 to ∂B_2 then allows computing an upper bound on the proportion s of ∂B_2 that is not contained in H. The ratio r/s then gives a lower bound on the number of facets of K_n. Unfortunately, even stating these results in detail would take more space than we have in this summary. We refer the reader to [BP01] for details, particularly sections §2 and §4.

3 Size of coefficients

By "size of coefficients" (Ziegler's title for [Zie97, §5]) we mean how large the size of an entry in the matrix A of the $\mathcal{H}-$presentation of a 0/1-polytope can be. To be more precise, every (full dimensional) 0/1-polytope P can be written a $P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$, where we may take A and $\mathbf{b}$ to have all integer entries, and each row $\mathbf{a}$ of A corresponds to a supporting hyperplane $H_{\mathbf{a}} = \{\mathbf{x} : \mathbf{ax} \leq \beta\}$ such that $H_{\mathbf{a}} \cap P$ is a facet of P. (We say that $\mathbf{a}$ is the *facet normal* vector of $H_{\mathbf{a}}$.) We can always write the inequality defining the hyperplane

$$\mathbf{ax} = a_1x_1 + \cdots + a_dx_d \leq \beta$$

uniquely such that the gcd of the coefficients $a_1, \ldots, a_d, \beta$ is 1. Under these assumptions, we say that the *greatest coefficient* of the inequality is $\max_{1\leq i\leq d} |a_i|$ The the greatest coefficient of P is the largest of the greatest coefficients over all of its facets (rows of A).

The greatest integer of a 0/1-polytope has practical importance in the design of optimizations algorithms, since for many such algorithms one of the steps is to obtain the $\mathcal{H}-$presentation of the polytope. There is a certain simplicity to 0/1-polytopes, specifically every vertex of a 0/1-polytope in $\mathbb{R}^d$ can be represented by a string of d bits, so the $\mathcal{V}-$presentation of such a polytope with n vertices has size $O(nd)$. One might naively hope that given this simplicity there might be some similarly nice bounds on the $\mathcal{H}-$presentation, implying a nice bound on the greatest coefficient. This turns out not to be the case. The one line summary : The greatest coefficient can be huge – doubly exponential in the dimension.

This fact has a close relationship with the arithmetics of 0/1-matrices and their inverses. Let ρ_n denote the maximum possible determinant of a $n \times n$ 0/1-matrix.

Lemma 3.1 (The Hadamard bound (Lemma 24 of Ziegler))**.** *The maximum determinant of a $n \times n$ 0/1-matrix is bounded by*

$$\rho_n \leq 2\left(\frac{\sqrt{n+1}}{2}\right)^{n+1}.$$

Aside: The corresponding result for a ± 1-matrix A of dimension $(n+1) \times (n+1)$ is $\det(A) \leq \sqrt{n+1}^{n+1}$. Those ± 1-matrices that attain this bound are known as *Hadamard matrices*. They have been the object of much study, and many open questions regarding them remain. Ziegler points to some literature regarding them. For our purposes, the Hadamard bound may be regarded as essentially sharp.

For an invertible $n \times n$ 0/1-matrix A, let $B = A^{-1}$, and define

$$\chi(A) := \max_{1\leq i,j\leq n} |b_{ij}|,$$

the largest absolute value of an entry in A^{-1}. Then define $\chi(n)$ to be the largest value of $\chi(A)$ taken over all invertible $n \times n$ 0/1-matrices A. Alon and Vũ [Zie97, Thm 25] prove bounds on $\chi(n)$, and this result together with Hadamard's bound is used to show [Zie97, Cor. 26]:

Corollary 3.2. *The largest integer coefficient* $\mathrm{coeff}(d)$ *in the facet description of a full-dimensional 0/1-polytope in $\mathbb{R}^d$ satisfies*

$$\frac{(d-1)^{(d-1)/1}}{2^{2d+o(d)}} \leq \chi(d-1) \leq \mathrm{coeff}(d) \leq \rho_{d-1} \leq \frac{d^{d/2}}{2^{d-1}}.$$

4 Related Topics

4.1 The Hirsch conjecture is theorem for 0/1-polytopes

A good discussion of the well-known Hirsch Conjecture is found in [Zie95, §3.3]. It makes a claim about the lengths of paths in the graph (1-skeleton) of a polytope.

Conjecture 4.1. *Let P be a d-dimensional polytope with n facets, and let $\boldsymbol{cx}$ be a linear function in general position with respect to P. Then there is a strictly increasing path with respect to $\boldsymbol{cx}$, from the (unique) vertex $\boldsymbol{v}_{\min}$ that minimizes $\boldsymbol{cx}$, from the (unique) vertex $\boldsymbol{v}_{\max}$ that maximizes $\boldsymbol{cx}$, of length at most $n-d$.*

This question has a bearing on certain approaches to optimization algorithms. The general conjecture is still open. No linear bound has been proved. However, the conjecture is known to be true for 0/1-polytopes.

Theorem 4.2 (Naddef)**.** *The diameter of the graph of a d-dimensional 0/1 polytope $P \subset \mathbb{R}^m$ is at most $\operatorname{diam}(G(P)) \leq d$, with equality if and only if P is (affinely equivalent to) a d-dimensional 0/1-hypercube. In particular, this implies that $\operatorname{diam}(G(P)) \leq n-d$, where n is the number of facets of P.*

4.2 Triangulations

A very natural question is how many simplices are needed to traingulate the d-dimensional 0/1-cube, assume we want proper triangulations which "fit together" in the normal sense of the phrase. If we define, as does Ziegler, $\operatorname{triang}(d)$ to be the smallest number of simplices needed to triangulate the d-dimensional 0/1-cube, then we have in particular that $\operatorname{triang}(7) = 1493$ [Zie97, §6.2]. This particular case is interesting since another result gives that for the 6-dimensional half-cube the minimal number of simplices required for a triangulation is precisely 1756. Since this number is greater than 1493, we have that not every d-dimensional 0/1-polytope has a triangulation into at most $\operatorname{triang}(d)$ simplices.

If we define

$$\sqrt[d]{\frac{\#\text{simplices}}{d!}}$$

to be the **efficiency** of a triangulation, then it was proven by Haiman [Hai91] that

$$L := \lim_{d\to\infty} \sqrt[d]{\frac{\operatorname{triang}(d)}{d!}}$$

exists, and as a result of Santos [San98] the best known upper bound

$$L \leq \sqrt[3]{\frac{7}{12}} \approx 0.836.$$

Ziegler says one would expect that the limit would be zero, so there is probably room for improvement here.

References

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