Lecturer: Rekha Thomas Due date: Monday, March 4, 2019, in class

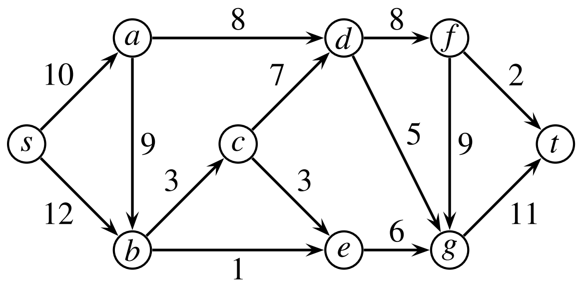
Problem Set 7

409 - Discrete Optimization

Winter 2019

Exercise 1

Consider the following network (G, u, s, t) (edges e are labelled with capacities $u(e)$):



a) Run the Ford-Fulkerson algorithm to compute a maximum s-t flow. After each iteration draw the current flow f and the corresponding residual graph G_f. What is the optimum flow value?

b) For the optimum flow f that you computed, define $S := \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$. Which are the nodes in S and what is the value $u(\delta^+(S))$ of the cut?

Exercise 2

In this exercise, you will give another proof of the Max-flow Min-Cut Theorem based on *Hoffman's Circulation Theorem.*

Let $G = (V, E)$ be a directed graph. A *circulation* on G is a function $f : E \to \mathbb{R}$ such that conservation of flow holds at *every* vertex $v \in V$. That is, a circulation must satisfy

$$\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$$

for every vertex $v \in V$.

Hoffman's Circulation Theorem states the following: Suppose $\ell : E \to \mathbb{R}$ and $u : E \to \mathbb{R}$ are functions that satisfy $\ell(e) \leq u(e)$ for every edge $e \in E$. Then there exists a circulation f on G satisfying

$$\ell(e) \leq f(e) \leq u(e)$$

for every edge $e \in E$ if and only if

$$\sum_{e \in \delta^-(A)} \ell(e) \leq \sum_{e \in \delta^+(A)} u(e)$$

for every set $A \subseteq V$.

Show that Hoffman's Circulation Theorem implies the Max-flow Min-cut Theorem. To be precise, you should prove that given a network $(G = (V, E), c, s, t)$ ($c(e)$ giving the capacity on e), there exists a flow of value equal to the minimum capacity k of a cut in the network. You do not need to reprove the fact that the maximum value of a flow is at most the minimum capacity of a cut.

Hint: Let G' be obtained from G by adding a new edge $e_0 = (t,s)$. (It is possible that e_0 runs in parallel to an existing edge in G; this poses no problem.) Define functions $\ell, u : E \to \mathbb{R}$ by $\ell(e) = 0$ and $u(e) = c(e)$ for $e \in E$ and $\ell(e_0) = u(e_0) = k$. Now apply Hoffman's Circulation Theorem to G' to argue that the original network G admits a flow of value k.