Lecturer: Rekha Thomas Due date: Monday, February 4, 2019, in class

Problem Set 4

# 409 - Discrete Optimization

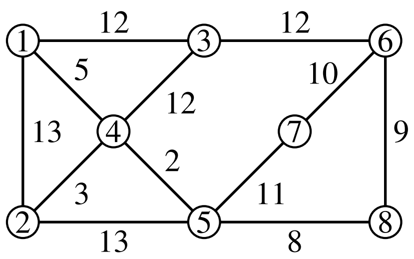
Winter 2019

**Exercise 1**

Consider the undirected graph $G = (V, E)$ with edge cost $c(e)$ for $e \in E$ that you can see below.

i) Compute a minimum spanning tree (it suffices to give the final tree).

ii) As you probably saw in i), edge $f = \{4,5\}$ is in the optimum tree. Now, let us imagine to change the cost of $f$ (but keep the other costs fixed). There is a threshold $\alpha$ so that whenever $c(f) < \alpha$ then any minimum spanning tree will contain $f$ and if $c(f) > \alpha$, then no minimum spanning tree will contain $f$. What is the value of $\alpha$?




**Exercise 2**

Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be vectors. We assume that $\text{span}(v_1, \ldots, v_m) = \mathbb{R}^n$. We call an index set $I \subseteq \{1, \ldots, m\}$ a *basis*, if the vectors $\{v_i\}_{i \in I}$ are a basis of $\mathbb{R}^n$. We assume that we are given cost $c(1), \ldots, c(m) \geq 0$ for all the vectors and abbreviate $c(I) := \sum_{i \in I} c(i)$ as the cost of a basis. We say that a basis $I^* \subseteq \{1, \ldots, m\}$ is *optimal* if $c(I^*) \leq c(I)$ for any basis $I$.

i) Let $I, J \subseteq [m]$ be two different basis. Prove that for all $i \in I \setminus J$, there is an index $j \in J \setminus I$ so that $(I \setminus \{i\}) \cup \{j\}$ is a basis and also $(J \setminus \{j\}) \cup \{i\}$ is a basis.
**Remark:** If you are unsure how to prove this, you may want to lookup *Steinitz exchange lemma* from your Linear Algebra course.

ii) Show that if a basis $I$ is not optimal, then there is an *improving swap*, which means that there is a pair of indices $i \in I$ and $j \notin I$ so that $J := (I \setminus \{i\}) \cup \{j\}$ is a basis with $c(J) < c(I)$.
**Remark:** The proof of this claim is actually along the lines of Theorem 4 on page 15 in the lecture notes. I recommend to read that proof before.

iii) We want to compute an optimum basis and we want to use the following algorithm:

(1) Set $I := \emptyset$

(2) Sort the vectors so that $c(1) \leq c(2) \leq \ldots \leq c(m)$

(3) FOR $i = 1$ TO $m$ DO

(4) If the vectors $\{v_j\}_{j \in I \cup \{i\}}$ are linearly independent, then update $I := I \cup \{i\}$

Prove that the computed basis $I$ is optimal.
**Remark:** Again, you might want to have a look into the correctness proof for Kruskal's algorithm in order to solve this.

**Exercise 3**
Prim's algorithm for finding a MST in $G = (V, E)$ is as follows:

1. Choose $v \in V$ and set $T := (\{v\}, \emptyset)$.
2. While $V(T) \neq V(G)$ do
   Choose $e \in \delta(T)$ of minimum cost. Update $T := T \cup \{e\}$.

Recall that for a collection of vertices $U \subset V$, $\delta(U)$ is the cut of $U$ consisting of all edges that have one endpoint in $U$ and the other endpoint outside $U$.

1. Find a MST in the graph from Exercise 1 using Prim's algorithm.
2. Call a forest $F$ in $G$ *greedy* if $F$ is contained in a MST of $G$. Suppose $F$ is a greedy forest and $U$ is one of its components and $e \in \delta(U)$. Prove that if $e$ has minimum cost among all edges in $\delta(U)$, then $F \cup \{e\}$ is again a greedy forest.
3. Use the above result to prove that Prim's algorithm is correct.