Math 409, Winter 2019 —- Problem Set 2

due Wednesday Jan 23, in class

Exercise 1

Consider the LP: max $c^\top x$ s.t. $Ax \leq b$. Assume that the feasible region $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polytope (bounded polyhedron). Prove that there is always a vertex of P that is optimal for this LP. (**Hint**: Recall the definition of a vertex of P from the notes. Argue that if the optimum is not a vertex, then we can write is as a convex combination of vertices of P, one of which must have the same or larger objective function value.)

Exercise 2

Consider a primal LP (P) and its dual LP (D). Justify which of the following nine cases are possible/impossible. You may use any of the theorems from Chapter 5.1.

1. (P) and (D) both have optimal solutions
2. (P) has an optimal solution and (D) is unbounded
3. (P) has an optimal solution and (D) is infeasible
4. (P) is unbounded and (D) has an optimal solution
5. (P) and (D) are unbounded
6. (P) is unbounded and (D) is infeasible
7. (P) is infeasible and (D) has an optimal solution
8. (P) is infeasible and (D) is unbounded
9. (P) and (D) are infeasible

Exercise 3

Here is a variant of Farkas lemma: $\exists\, x : Ax \leq b \;\Leftrightarrow\; \nexists\, y \geq 0, y^\top A = 0, y^\top b < 0$.

In this exercise we will use this Farkas lemma to show that every feasible bounded LP has an optimal solution. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be the polyhedron that is the feasible region of the LP. Since the LP is bounded, there is some δ such that $\delta = \sup\,\{c^\top x : x \in P\}$. To argue that there is an optimal solution x to the LP, we will show that there is an $x \in P$ such that $c^\top x \geq \delta$. Then it will follow that $c^\top x = \delta$ and this x would be the optimal solution of the LP.

Use the above variant of Farkas lemma to argue that there is an $x : Ax \leq b$ and $c^\top x \geq \delta$.

Hints:

1. Suppose $\nexists x : Ax \leq b,$ and $c^\top x \geq \delta$. Then the above version of the Farkas lemma says that there is a y such that something happens. Write down what this is.
2. Argue that the component of y that multipies $c^\top$ can be assumed to be 1.
3. Derive a contradiction to one of the duality theorems.

Exercise 4

1. Let $G = (V, E)$ be an undirected graph. A *stable set* in G is a subset of vertices $S \subseteq V$ such that for any two vertices i and j in S, the pair $\{i, j\}$ is NOT an edge in G. Model the problem of finding the largest stable set in G as an integer linear program.

2. Construct a graph on 5 vertices with at least 5 edges (but not all possible edges) and circle the vertices that form a stable set.

3. Let $G = (V, E)$ be an undirected graph. A *clique* in G is a subset of the vertices $K \subseteq V$ such that for any two vertices i and j in K, the pair $\{i, j\}$ is an edge in G. Circle the vertices of a clique in your graph from part 2.

4. The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ on the same vertex set V whose edges are precisely the missing edges of G. More precisely,

$$\overline{E} = \{\{i, j\} : i, j \in V, \ \{i, j\} \notin E\}.$$

   Draw the complement of your graph from part 2.

5. What is the relationship between stable sets in G and cliques in $\overline{G}$? Justify

6. Model the problem of finding the largest clique in G as an integer program using the above ideas.