# Math 409, Winter 2019 —- Problem Set 1

due Monday Jan 14, in class

**Exercise 1**

Write down the definition of a convex set and use it to prove that polyhedra are convex sets.

**Exercise 2**

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full-dimensional polyhedron in $\mathbb{R}^n$ and $p$ a point in $P$. Let $A'x \leq b'$ be the subset of inequalities from $Ax \leq b$ that hold at equality at $p$. This means that $A'p = b'$ and $a_i^\top p < b_i$ if $a_i{}^\top$ is a row of $A$ that is not a row of $A'$. Prove that if $p$ is a vertex of $P$, then $\text{rank}(A') = n$.

**Exercise 3**

Do all polyhedra have vertices? If yes, prove it. If no, find a counterexample.

**Exercise 4**

Here is a variant of Farkas lemma: $\exists\, x : Ax \leq b \;\Leftrightarrow\; \nexists\, y \geq 0, y^\top A = 0, y^\top b < 0.$

In this exercise we will use this Farkas lemma to show that every bounded LP has an optimal solution. Let $P$ be the polyhedron that is the feasible region of the LP. Since the LP is bounded, there is some $\delta$ such that $\delta = \sup\,\{c^\top x : x \in P\}$. To argue that there is an optimal solution $x$ to the LP, we will show that there is an $x \in P$ such that $c^\top x \geq \delta$. Then it will follow that $c^\top x = \delta$ and this $x$ would be the optimal solution of the LP.

Use the above variant of Farkas lemma to argue that there is an $x : Ax \leq b$ and $c^\top x \geq \delta$.

**Exercise 5**

1. Write down the dual of the following LP:

$$\max\,\{x_2 : x_1 + x_2 \leq 3,\; x_1 + 2x_2 \leq 4,\; x_1 - x_2 \geq -1\}.$$

2. Solve the primal LP using geometry and write down the optimal solution.

3. In the proof of strong duality, we saw that there is a geometric way to find the optimal solution of the dual LP from the optimal solution of the primal LP. Use this to find the dual optimal solution.

4. How can you use strong duality to justify the correctness of your dual optimal solution?