Lecturer: Rekha Thomas Due date: Monday, May 14, 2018, in class

Problem Set 6

409 - Discrete Optimization

Spring 2018

Exercise 1

Here is a variant of Farkas lemma: $\exists\, x : Ax \leq b \;\Leftrightarrow\; \nexists\, y \geq 0, y^\top A = 0, y^\top b < 0$.

In this exercise we will use this Farkas lemma to show that every bounded LP has an optimal solution. Let P be the polyhedron that is the feasible region of the LP. Since the LP is bounded, there is some δ such that $\delta = \sup\,\{c^\top x : x \in P\}$. To argue that there is an optimal solution x to the LP, we will show that there is an $x \in P$ such that $c^\top x \geq \delta$. Then it will follow that $c^\top x = \delta$ and this x would be the optimal solution of the LP.

Use the above variant of Farkas lemma to argue that there is an $x : Ax \leq b$ and $c^\top \geq \delta$.

Exercise 2

Consider the LP: max $c^\top x$ s.t. $Ax \leq b$. Assume that the feasible region $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polytope (bounded polyhedron). Prove that there is always an extreme point (vertex) of P that is optimal for this LP.
(**Hint**: Recall the definition of an extreme point of P from the notes. Argue that if the optimum is not an extreme point, then we can write is as a convex combination of extreme points of P, one of which must have the same or larger objective function value.)

Exercise 3

Model the minimum spanning tree problem as an integer program.

Exercise 4

Let $G = (V,E)$ be a bipartite graph with parts $V = V_1 \dot\cup V_2$. Consider the linear program:

$$\begin{aligned} \min \sum_{u \in V} y_u & & & \\ y_u + y_v &\geq& 1 & \quad \forall \{u,v\} \in E \\ y_u &\geq& 0 & \quad \forall u \in V \end{aligned}$$

a) If you write the problem in the matrix form $\min\{\mathbf{1}^T y \mid Ay \geq \mathbf{1}; y \geq \mathbf{0}\}$, how is the matrix A defined?

b) Prove that all extreme points of $P = \{y \in \mathbb{R}^V \mid Ay \geq \mathbf{1}; y \geq \mathbf{0}\}$ are integral (A defined as in a)).

c) Which problem that you know from the lecture, does the above LP solve?