Lecturer: Rekha Thomas Due date: Wednesday, April 25, 2018, in class

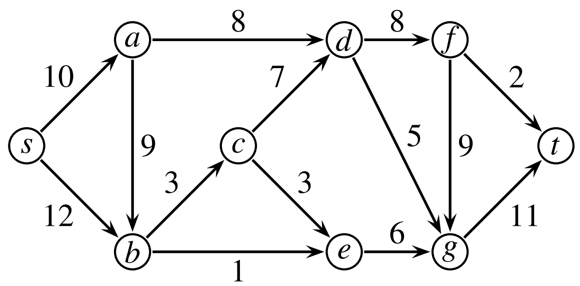
Problem Set 4

409 - Discrete Optimization

Spring 2018

Exercise 1

Consider the following network (G, u, s, t) (edges e are labelled with capacities $u(e)$):

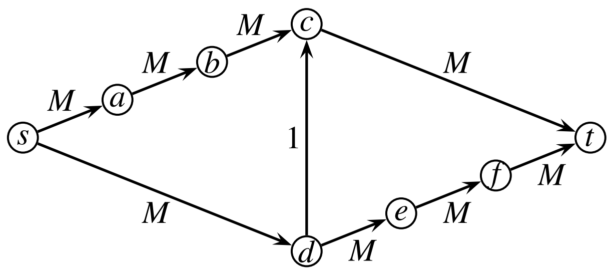


a) Run the Ford-Fulkerson algorithm to compute a maximum s-t flow. After each iteration draw the current flow f and the corresponding residual graph G_f. What is the optimum flow value?

b) For the optimum flow f that you computed, define $S := \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$. Which are the nodes in S and what is the value $u(\delta^+(S))$ of the cut?

Exercise 2

Consider the following network with a directed graph $G = (V, E)$, capacities $u(e)$ (the labels of the edges), a source s and a sink t (assume that $M > 1$).



a) Argue that the Ford-Fulkerson algorithm with a poor choice of augmenting paths might take M or more iterations.

b) Run the Edmonds-Karp algorithm on this network and give the flow in each iteration.

Exercise 3

In this exercise, you will give another proof of the Max-flow Min-Cut Theorem based on *Hoffman's Circulation Theorem.*

Let $G = (V, E)$ be a directed graph. A *circulation* on G is a function $f : E \to \mathbb{R}$ such that conservation of flow holds at *every* vertex $v \in V$. That is, a circulation must satisfy

$$\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$$

for every vertex $v \in V$.

Hoffman's Circulation Theorem states the following: Suppose $\ell : E \to \mathbb{R}$ and $u : E \to \mathbb{R}$ are functions that satisfy $\ell(e) \leq u(e)$ for every edge $e \in E$. Then there exists a circulation f on G satisfying

$$\ell(e) \leq f(e) \leq u(e)$$

for every edge $e \in E$ if and only if

$$\sum_{e \in \delta^-(A)} \ell(e) \leq \sum_{e \in \delta^+(A)} u(e)$$

for every set $A \subseteq V$.

Show that Hoffman's Circulation Theorem implies the Max-flow Min-cut Theorem. To be precise, you should prove that given a network $(G = (V, E), c, s, t)$ ($c(e)$ giving the capacity on e), there exists a flow of value equal to the minimum capacity k of a cut in the network. You do not need to reprove the fact that the maximum value of a flow is at most the minimum capacity of a cut.

Hint: Let G' be obtained from G by adding a new edge $e_0 = (t, s)$. (It is possible that e_0 runs in parallel to an existing edge in G; this poses no problem.) Define functions $\ell, u : E \to \mathbb{R}$ by $\ell(e) = 0$ and $u(e) = c(e)$ for $e \in E$ and $\ell(e_0) = u(e_0) = k$. Now apply Hoffman's Circulation Theorem to G' to argue that the original network G admits a flow of value k.

Exercise 4

Let (G, u, s, t) be a network with $n = |V|$ nodes and $m = |E|$ edges and $u(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$. Suppose that f^* is the optimum max-flow. In this exercise, we want to develope a faster version of the Ford-Fulkerson algorithm. In fact, we want to modify the algorithm so that in each iteration the algorithm chooses the path P that maximizes the bottleneck capacity $\gamma = \min\{u_f(e) \mid e \in P\}$. We call that algorithm "smart FF".

a) Show that in the first iteration, smart FF finds already a flow f with $\text{val}(f) \geq \frac{1}{2m}\text{val}(f^*)$.
Hint: The claim says essentially that even after we delete all edges e that have small capacity, say $u(e) < \frac{1}{2m}\text{val}(f^*)$, the network will not become disconnected. It might be helpful to remember the MaxFlow=MinCut Theorem.

b) Now suppose we already computed some s-t flow f. Show that there exists a flow g in G_f with $\text{val}(g) \geq \text{val}(f^*) - \text{val}(f)$.
Hint: This is somewhat the reverse process of augmenting a flow.

c) We want to generalize the claim in a). Consider any iteration of smart FF and say that f is the flow that we computed so far. Show that there exists always a path P in G_f on which the bottleneck capacity $\min\{\mu_f(e) \mid e \in P\}$ is at least $\frac{1}{2m}(\text{val}(f^*) - \text{val}(f))$.

d) Show that smart FF needs at most $O(m \cdot \log(\text{val}(f^*))$ many iterations. **Hint:** Suppose that $f_0, f_1, \ldots, f_T$ is the sequence of flows that we compute in T iterations. Argue that after t iterations, our flow has a value of at least $\text{val}(f^*) \cdot (1 - (1 - \frac{1}{2m})^t)$.