MATH 409
HW 5 SOLUTIONS

Solution to Exercise 2, Lecture 13. The LP formulation is given as

$$
\begin{aligned}
\max: & \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e \\
\text{s.t.}: & \sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e = 0 && \forall v \in V(G)\setminus\{s,t\} \\
& 0 \le x_e \le u_e && \forall e \in E(G).
\end{aligned}
$$

We are going to translate this into *canonical form* so that we can easily write down the dual. First, we order the edges and vertices of G, say $V = (v_1, \ldots, v_n)$ and $E = (e_1, \ldots, e_m)$. Let B be the *incidence matrix* of the directed graph G; by definition, this is the $n \times m$ matrix $B = (b_{ij})$ where

$$
b_{ij} = \begin{cases} 1 & \text{if } e_j \in \delta^+(v_i) \\ -1 & \text{if } e_j \in \delta^-(v_i) \\ 0 & \text{otherwise.} \end{cases}
$$

Suppose that $s = v_i$ and $t = v_j$. Let A be the $(n-2)\times m$ matrix obtained from deleting the ith and jth rows from B, and let c^T denote the ith row of B.

Corresponding to the flow f (or $\{x_e\}$), and the capacity u we define the vectors

$$
x := \begin{pmatrix} x_{e_1} \\ \vdots \\ x_{e_m} \end{pmatrix} \qquad u := \begin{pmatrix} u_{e_1} \\ \vdots \\ u_{e_m} \end{pmatrix}.
$$

Observe the following:

$$
\sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e = c^T x
$$

$$
\sum_{e\in\delta^+(s)} x_e - \sum_{e\in\delta^-(s)} x_e = 0 \quad \forall v \in V(G)\setminus\{s,t\} \iff Ax = 0
$$

$$
0 \le x_e \le u_e \forall e \in E(G) \iff x \le u.
$$

So the canonical form for the LP is given by

$$\begin{aligned} \max : & \, c^T x \\ \text{s.t. :} & \begin{pmatrix} A \\ -A \\ I \end{pmatrix} x \le \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \\ & 0 \le x. \end{aligned}$$

Then the dual LP is:

$$\begin{aligned} \min : & \begin{pmatrix} 0 & 0 & u^T \end{pmatrix} y \\ \text{s.t. :} & \begin{pmatrix} A^T & -A^T & I \end{pmatrix} y \ge c \\ & 0 \le y. \end{aligned}$$

What follows is not part of what was graded; it was not required for the homework. We can interpret this dual LP in terms of the original graph. First check that y is a column vector of size $(n-2)+(n-2)+m$; write $y^T = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$, where α, β and γ are vectors of the corresponding size. Then we will associate the kth entry $\gamma(k)$ with the edge e_k in G, and the kth entry of α and β with v_k (skipping $s = v_i$ and $t = v_j$).

The dual LP has objective min $\sum_k u_{e_k} \gamma(k)$; in the case where γ represents some choice function $\gamma : E(G) \to \{0, 1\}$, then the objective minimizes the total capacity of the chosen edges. The fact is that a certain 0-1 integer program associated with the dual LP solves the problem of calculating the minimal s-t cut, i.e. it produces an s-t cut with minimal total capacity. We will not prove this now, but we can say a little bit more.

Since each column of A has only two non-zero entries, we can easily compute $A^T\alpha$, $-A^T\beta$, and $I\gamma$; each row in the matrix inequality on y is associated with some edge $e = v_k v_l$; for this edge, the inequality translates as

$$(1-\delta_{ik})\,[\alpha(k) - \beta(k)] - (1-\delta_{jl})\,[\alpha(l) - \beta(l)] + \gamma(e) \ge \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } l = i \\ 0 & \text{otherwise.} \end{cases}$$

Observe that we can reduce our program by defining $p(k) := \alpha(k) - \beta(k)$ for all $k \in V(G) \setminus \{s, t\}$ (since α and β always appear as $\alpha - \beta$ in the constraint inequality, and do not appear in the objective). The function p is often called the *potential* in other references. □

Solution to Exercise 3, Lecture 13.

$$\begin{aligned} \min : & \sum_{e \in E} c_e x_e \\ \text{s.t. :} & \sum_{e=(i,j),\ i,j \in S} x_e \leq |S| - 1 \qquad \forall S \subseteq V(G) \\ & \sum_e x_e = |V(G)| - 1 \\ & 0 \leq x_e \leq 1. \end{aligned}$$

The associated integer program has $x_e \in \{1, 0\}$. Suppose x satisfies the conditions (but perhaps is not minimal); let $T = \{e \mid x_e = 1\}$. We will show that T is a tree. Observe that T consists of $|V(G)| - 1$ edges; so it will suffice to show that T is acyclic. Assume, for contradiction, that T contains a cycle of length k. We can further assume that this is the smallest length cycle in T. Then the k vertices incident the edges in this cycle form a set S such that $\sum_{e=(i,j),\ i,j \in S} x_e = |S| = k$, contradicting our choice of x. Thus T is a tree. Conversely, if T is a tree, we can define

$$x_e := \begin{cases} 1 & \text{if } x_e \in T \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that x satisfies the conditions. Since the objective function is exactly the weight of a spanning tree, minimizing the objective function is exactly finding a minimal spanning tree. □

Solution to Exercise 5, Lecture 14. I will describe the f-augmenting paths:

$$\begin{aligned} s - p - b - t & \qquad \gamma = 1 \\ s - q - a - t & \qquad \gamma = 1 \\ s - q - a - p - b - t & \qquad \gamma = 1 \\ s - q - b - t & \qquad \gamma = 1. \end{aligned}$$

There is no other f-augmenting path, so the maximum flow is 4. Then a minimal cut can be found by taking $A = \{s, q, a\}$; indeed, $\delta(A) = \{sp, qb, at, ap\}$ is a cut of weight 4, which shows that both the cut and the flow are optimal. □

Solution to Exercise 5, Lecture 16. (i) Observe that $\delta'(R) \cap E = \delta(A)$ is a finite capacity cut of G', which implies that it contains no edges of E. Since $\delta(A) \subset E$, we must have $\delta(A) = \emptyset$, so that A is a closure of V.

(ii) We know that $\delta(A) = \emptyset$, so if $R = \{s\} \cup A$ as above,

$$\delta'(R) = \{sv \mid b_v > 0, v \notin A\} \cup \{vt \mid b_v < 0, v \in A\}.$$

It follows that the capacity is

$$\sum \{b_v \mid b_v > 0, v \notin A\} - \sum \{b_v \mid b_v < 0, v \in A\}.$$

(iii) Observe that

$$\begin{aligned} \delta'(R) &= \sum \{b_v \mid b_v > 0, v \notin A\} + \sum \{b_v \mid b_v \geq 0, v \in A\} \\ &\quad - \sum \{b_v \mid v \geq 0, v \in A\} - \sum \{b_v \mid b_v < 0, v \in A\} \\ &= \sum \{b_v \mid v \in V, b_v \geq 0\} - \sum \{b_v \mid v \in A\} \\ &= \sum \{b_v \mid v \in V, b_v \geq 0\} - b(A). \end{aligned}$$

Since $\sum \{b_v \mid v \in V, b_v \geq 0\}$ is independent of A, we can see directly that maximizing $b(A)$ corresponds to minimizing $\delta'(R)$, as desired.

(iv) The solution is $\{d, e, f, g\}$ with optimal benefit 1. □

Solution to Exercise 4, Lecture 17.

	Maximal Matching LP		Minimal Cover LP
max :	$\sum_{e \in E} x_e$	min :	$\sum_{v \in V} x_v$
s.t. :	$\sum_{e \in E \text{ s.t. } v \in e} x_e \leq 1$	s.t. :	$x_i + x_j \geq 1 \quad \forall e = (i, j) \in E(G)$
	$0 \leq x_e \leq 1.$		$0 \leq x_v \leq 1.$

Maximal Matching. The variables x_e indicate whether the edge e is in the matching or not. Being a matching means that each vertex is incident to at most one edge; in other words, the sum over the edges incident to v is ≤ 1 for each v.

Minimal Cover LP. The variables x_v indicate whether the vertex v is in the cover. To be a cover, we must have that any edge $e = (i, j)$ is incident to some vertex in the cover, so that $x_i = 1$ or $x_j = 1$; equivalently, if $e = (i, j)$ then $x_i + x_j \geq 1$.

These two problems ARE related to each other, but not naively. They are only dual linear programs when the associated graph is bipartite. □

Solution to Exercise 7, Lecture 17. Let $R := \{a_1, \ldots, a_m\}$ and $C := \{b_1, \ldots, b_n\}$ be the set of rows and columns of the matrix M. Define a complete bipartite graph $R \cup C$. We will show how a matrix M, with no two nonzero entries in the same row, determines a matching: we admit an edge $e = a_i b_j$ in a matching if there is a nonzero entry in position ij in M, and conversely. The fact that no two nonzero entries occur in the same row or column verifies that no two edges in M are incident to the same vertex.

Recall that a cover is a set of vertices such that each edge e is incident to at least one $c \in C$. Since the vertices are just rows or columns, this is a set of rows and columns containing all non-zero entry.

By König's lemma, the size of a maximal matching is the size of a minimal cover; thus the most nonzero entries we can place in M such that no two are in a row is equal to the smallest number of lines containing all of the nonzero entries. (In both cases, this is just $\min\{m, n\}$.) □