MATH 409
HW 8 EXERCISES

Solution to Exercise 1.

(i) Note that we are minimizing. Observe that at branch 7, we are given that $z \leq 31$, which is the smallest of the upper bounds given. Similarly, at branch 5 we have that $z \geq 27$, which is the smallest of the lower bounds given. Thus the optimal value satisfies $27 \leq \bar{z} \leq 31$.

(ii) Branch 8 can be pruned because it is infeasible. Branch 7 can be pruned because its optimal value is 31. Branch 8 can be pruned because its lower bound is greater than the upper bound at 7.

We must explore the branches at 5 and 3. We should explore 5 first, since its smallest possible optimum value is less than that at 3. □

Solution to Exercise 2. We find the following (nonzero) vertices for the polyhedron:

$$(1,0), \quad (\frac{59}{11}, \frac{16}{11}), \quad (6, \frac{1}{2}), \quad (6,0).$$

Since the objective function f has positive coefficients, we only need to compare $f(6,1/2) = 56.5$ with $f(59/11, 16/11) = 55.5$. Thus, the maximum occurs at $(6, \frac{1}{2})$; we then branch at $x_2 = 1/2$, creating two new polyedrons S_1 and S_2. The first has the additional constraint $x_2 \geq \lceil 1/2 \rceil = 1$, and the second has the constraint $x_2 \leq \lfloor 1/2 \rfloor = 0$.

We calculate the vertices of S_1 as $(4,1)$, $(17/3, 1)$, $(59/11, 16/11)$, with objective values 41, 56, and 55.545, respectively. Thus the optimum value 56 occurs at $(17/3, 1)$.

We calculate the vertices of S_2: $(1,0)$ and $(6,0)$. Thus the optimum occurs at $(6,0)$ and is 54. Since $(6,0)$ is integral, we do not need to further explore this branch.

Since the optimum value of S_1 is greater than the optimum value of S_2, we need to continue exploring S_1. So we branch S_1 at $x_1 = 17/3$, creating two new polyedrons S_{11} an S_{12}, with additional constraints $x_1 \geq \lceil 17/3 \rceil = 6$, and $x_1 \leq \lfloor 17/3 \rfloor = 5$. Observe that S_{11} is infeasible, so we prune this branch. We calculate the vertices of S_{12} as $(4,1)$, $(5,1)$ and $(5, 4/3)$, with objective values 41, 50, and 51.67. The optimum occurs at $(5, 4/3)$; however, the optimum value is 51.67 is less than the optimum value of S_2; thus we can prune S_{12}.

This proves that the integer program has optimum $(6, 0)$ with optimal value 54. □

Solution to Exercise 3.

(i) The basic idea is that we can rescale the x_is and obtain an equivalent linear program with an obvious solution. So let $y_i = a_i x_i$; and let $d_i = c_i/a_i$. Then the objective function $\sum_i d_i y_i$ is equal to $\sum_i a_i x_i$; and similarly

$$\sum_{i=1}^{n} a_i x_i \le b \iff \sum_{i=1}^{n} y_i \le b.$$

Also, the constraint $x_i \le 1$ is equivalent to $y_i \le a_i$.

Now suppose that $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, and observe that the cost of each y_i is the same (it is 1). Thus, we cannot do better than including as much of y_1 as possible, since it has the highest profit. So we set $y_1 = \min\{a_1, b_1 := b\}$.

We have $b_2 := b_1 - a_1$ left to "spend," so as before we set $y_2 = \min\{a_2, b_2\}$. We proceed inductively. Observe that $b_i = b - \sum_{i=1}^{i-1} a_i$. Then if r is the smallest integer such that $b_{r+1} < 0$, then $\min\{a_i, b_i\} = a_i$ for $1 \le i \le r-1$, and $\min\{a_i, b_r\} = b_r$. Thus the optimum solution is $y_i = a_i$ for $i = 1, \ldots, r-1$, $y_r = b_r = b - \sum_{i=1}^{r-1} a_i$, and $y_j = 0$ for $j > r$.

This means that the equivalent solution $x_i = 1$ for $i = 1, \ldots, r-1$, $x_r = \frac{b - \sum_{i=1}^{r-1} a_i}{a_r}$, and $x_j = 0$ for $j > r$ is optimal for the original problem.

(ii) We solve this using the branch and bound method, knowing from the previous part a very quick method for finding the solution to (each) LP relaxation. Let S be the LP relaxation of the problem. Then choosing $r = 3$ the solution for S is $1 = x_1 = x_2$, $x_3 = \frac{12-8}{8} = 1/2$, $x_4 = 0$ with optimal value 39.5. So we branch at $x_3 = 1/2$. Let S_1 and S_2 be the problems given by the additional constraints $x_3 \le 0$ and $x_3 \ge 1$. In particular, this means that $x_3 = 0$ or $x_3 = 1$.

We first explore S_1: $(x_3 = 0)$ the constraint equation is

$$5x_1 + 3x_2 + 7x_4 \le 12.$$

We still have the condition that $c_1/a_1 \ge c_2/a_2 \ge c_4/a_4 > 0$. So choosing $r = 3$ gives the optimum $1 = x_1 = x_2$, and $x_4 = 4/7$, with optimal value 36.71.

Now we explore S_2: $(x_3 = 1)$ the constraint equation is

$$5x_1 + 3x_2 + 7x_4 \leq 4.$$

So choosing $r = 1$, we have the optimum at $x_1 = 4/5$, $x_2 = x_4 = 0$ with optimal value 38.6.

None of these optimums are integral solutions, so we must branch again. We first explore S_{11}: $(x_4 = 0 = x_3)$. The constraint equation is

$$5x_1 + 3x_2 \leq 12.$$

Choosing $r = 3$, we get the optimum $1 = x_1 = x_2$, with optimal value 27.

Now we explore S_{12}: $(x_4 = 1, x_3 = 0)$. The constraint is

$$5x_1 + 3x_2 \leq 5.$$

Choosing $r = 2$, we find the optimum $x_1 = 1$ and $x_2 = 0$, with optimal value 34.

S_{21}: $(x_1 = 0, x_3 = 1)$. The constraint equation is

$$3x_2 + 7x_4 \leq 4.$$

Choosing $r = 2$, we have the optimum at $1 = x_2$ and $x_4 = 1/7$, with optimal value 37.4.

S_{22}: $(x_1 = 1, x_3 = 1)$. The constraint equation is

$$3x_2 + 7x_4 \leq -1.$$

This is not feasible, so we prune this branch.

We still must explore S_{21}, since its optimal value is greater than any of the integral optimal values obtained.

S_{211}: $(x_4 = 0, x_1 = 1, x_3 = 1)$. The constraint equation is

$$3x_2 \leq 4.$$

This has optimum at $x_2 = 1$, with optimal value 35.

S_{212}: $(x_4 = 1, x_1 = 1, x_3 = 1)$. The constraint equation is

$$3x_2 \leq -3.$$

This is not feasible, so we prune this branch.

So now we compare the optimal values, and we see that $(0, 1, 1, 0)$ is the solution to the integral problem with optimal value 35. □

Solution to Exercise 4. There are three vertices defining the minimal boundary of the unbounded polytope, namely

$$(0, 4), \quad (15/4, 1/4), \quad (5, 0).$$

These have objective values 8, 17/4 and 5, respectively. Thus the optimum 17/4 (the minimum) occurs at $x^* := (15/4, 1/4)$.

The idea next is to find an additional constraint which every integer solution must satisfy, but one which $(15/4, 1/4)$ does not satisfy. Observe that our constraints can be re-written as

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 \geq 2$$
$$\frac{1}{2}x_1 + \frac{5}{2}x_2 \geq \frac{5}{2}.$$

If we add these together, we obtain

$$x_1 + 3x_2 \geq \frac{9}{2}.$$

If x_1, x_2 are integers, then $x_1 + 3x_2 \geq 9/2$ implies that $x_1 + 3x_2 \geq 5$. Thus we impose this additional constraint. Now observe that

$$\frac{15}{4} + 3 \cdot \frac{1}{4} = 9/2 < 5,$$

so that x^* does not satisfy the new constraint. □