MATH 409
HW 7 EXERCISES

Solution to Exercise 2, Lecture 22. Square submatrices M of A correspond bijectively with square submatrices M^T of A^T. We know from linear algebra that $\det(M^T) = \det(M)$; thus, requiring that all square submatrices M of A have determinant $0, \pm 1$ is the same as requiring that all square submatrices of M^T have determinant ± 1, i.e. A is TU (*totally unimodular*) iff A^T is TU.

It is easy to see that

$$(A, -A, I) \text{ is } TU \implies (A, I) \text{ is } TU \implies A \text{ is } TU,$$

since every square submatrix of (A, I) is in particular a square submatrix of $(A, -A, I)$ and similarly for A. Thus it remains to show that A is $TU \implies (A, -A, I)$ is TU. We isolate two facts to prove this, since they are of general interest.

Proposition 1. *Suppose* $A = [a_1 \cdots a_n]$ *is an* $m \times n$ TU *matrix, where each* a_i *is a column vector; then*

(i) *The matrix* $[A \mid \pm a_i]$ *is also* TU.
(ii) *If* v *is an* $m \times 1$ *column vector, where every entry is zero except* $v(i) = \pm 1$, *then* $[A \mid v]$ *is* TU.

Remark. Similar statements apply to rows (instead of columns) by taking the transpose. Also, in (ii) we can take v to be the 0 vector (i.e. such that $v(i) = 0$ as well). This is trivial.

We will prove this proposition next. Now we show how it solves our problem: let $A_1 := [A \mid -a_1]$. By (i), A_1 is TU; similarly, $A_2 := [A_1 \mid -a_2]$ is TU, and so inductively, $A_n = [A \mid -A]$ is TU. Then write $I_{m\times m} = [v_1 \cdots v_m]$. By (ii), $B_1 := [A_n \mid v_1]$ is TU; and again, $B_2 := [B_1 \mid v_2]$ is TU, and inductively $B_m := [A_n \mid I_{m\times m}]$ is TU. Since $B_m = [A \mid -A \mid I_{m\times m}]$, this completes the proof. It remains to verify the two facts above.

Proof.

(i) Let $v := a_j$; we will show that the matrix $C := [A \mid \pm v]$ is again TU. To prove this, consider any square submatrix M of C. There are a few cases: if M does not contain entries from column v, then M is a square submatrix of A and thus $\det(M) \in \{\pm 1, 0\}$. Otherwise, M contains entries from the (last) column $\pm v$. If M also contains entries from a_j, then these two columns in M have to be identical and thus $\det(M) = 0$ since the column space does not have full rank. The last case is that M contains the entries from $\pm v$, but not from a_i. By multiplying $-v$ by -1 if necessary, and permuting the columns of M we obtain M', some square submatrix of A (we need to move v to the left some number of columns). But from linear algebra, we know that multiplying a column by -1 and permuting the columns of a matrix only affects the determinant by a sign change; thus $\det(M) = \pm\det(M') \in \{\pm 1, 0\}$. Since an arbitary square submatrix of C has determinant in $\{\pm 1, 0\}$, we have that C is TU.

(ii) Consider the matrix $C := [A \mid v]$; let M be any submatrix of C. If M does not contain v, then M is a square submatrix of A and so $\det(M) \in \{\pm 1, 0\}$ since A is TU. Otherwise, M contains v, and we can expand the determinant of M along v "by minors". If M is $k \times k$, then $\det(M) = \pm\det(M_{ik})$; but M_{ik} is a square submatrix of A (we have crossed out the column v and the row i), and thus $\det(M) = \pm\det(M_{ik}) \in \{\pm 1, 0\}$ since A is TU. □

□

Solution to Exercise 6, Lecture 22. We have that

$$A_1 := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we delete first and second columns and the last row, we obtain the square submatrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which has determinant 2. Thus A_1 is not TU. Alternatively, you can recognize A_1 as the incidence matrix of an undirected graph, namely the complete graph K_4 minus one edge. Since this graph is not bipartite (it has a cycle of length 3), Proposition 5 (2) shows that A_1 is not TU.

We will now show that

$$A_2 := \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

is TU by verifying Proposition 3 of Lecture 22. Here, we let $M_1 = \{r_1, r_2\}$, and $M_2 = \{r_3, r_4, r_5\}$. Every column contains at most two non-zero entries, and one can easily check that in a given column the sum over M_1 is the same as the sum over M_2, as required. □

Solution to Exercise 8, Lecture 22. We define the following matrices:

$$A := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad x := \begin{pmatrix} y \\ x_1 \\ \vdots \\ x_m \end{pmatrix} \qquad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then it is easy to see that $Ax \leq b$, $x \geq 0$ is exactly the restrictions on the points in the polyhedron P. Since b is integral, to show that $P = P^I$ it is sufficient to show that A is TU by Theorem 7 (ii) of Lecture 22. Let $A = [a_1 \cdots a_{m+1}]$, where a_i is the ith column in A above. Observe that $[a_1]$ is TU. Since a_i, $i > 1$ is a column vector of 0's except for one 1, we may apply proposition 1 (ii) above; this says that $[a_1\, a_2]$ is TU, and thus $[a_1\, a_2\, a_3]$ is TU, and so on. Inductively, we see that A is TU, as desired. □