MATH 409
WEEK 6 EXERCISES

Solution to Exercise 8, Lecture 17. One direction is obvious, because given a matching of size $|P|$ every vertex in P must be incident to some edge in the matching. Thus if $A \subseteq P$, $N(A)$ contains at least the $|A|$ distinct neighbors of A which are neighbors via the matching. So $|N(A)| \geq |A|$.

The converse is trickier. Remember we're assuming $|N(A)| \geq |A|$ for all $A \subseteq P$ and we want to show that there is a matching of size $|P|$. Let M be a maximal matching and let C be a minimal cover; by König's theorem, $|C| = |M|$, so it is sufficient to show that $|C| = |P|$. (Indeed, a matching of size P would have to be maximal, so we have not made the problem any more difficult.) Furthermore, P is already a covering, so we know that $|C| \leq |P|$.

Since C is a cover, each edge in G is incident to some vertex in C. So there are no edges between $P \setminus C$ and $Q \setminus C$. Thus the neighbors of $P \setminus C$ are in $Q \cap C$, and so if $A := P \setminus C$, since $|A| \leq |N(A)|$ we see that $|P \setminus C| \leq |Q \cap C|$. Then observe that

$$|P| = |P \setminus C| + |P \cap C| \leq |Q \cap C| + |P \cap C| = |C|. \qquad \square$$

Solution to Exercise 4, Lecture 19-21. Note that we need that G does not contain loops. we proceed by induction on $n = |V(G)|$. The induction hypothesis is that for every graph G such that $|V(G)| \leq k$, the given algorithm colors the vertices such that at least half of the edges are incident to vertices of opposite color.

Base case, $k = 2$: The algorithm colors one vertex red and the other blue, so all edges are incident to vertices of opposite color.

Induction step: consider G such that $|G| = k + 1$. Choose a vertex $v \in G$, and consider the graph $G' := G \setminus \{v\}$. Since $|V(G')| = k$, we may apply the induction hypothesis to color the vertices such that half of the edges are incident to vertices of opposite color. Then apply the algorithm to the final vertex: count the number r_v of edges incident to v which are incident to a red vertex, and the number b_v of edges incident to v which are incident to a blue vertex. If $b_v \geq r_v$ then we color v blue, and otherwise red. This means that at least half of the new edges will be between vertices of opposite color; so at least half of all the edges are between vertices of opposite color, as required. $\square$

Solution to Exercise 5, Lecture 19-21. The coefficients of the objective function and the constraint inequality are, respectively,

$$c_1 = 3, \quad c_2 = 2, \quad c_3 = 1, \quad c_4 = 1 \quad c_5 = 1$$
$$w_1 = 1, \quad w_2 = 2, \quad w_3 = 3, \quad w_4 = 4, \quad w_5 = 5.$$

Then $n = 5$, $C = 8$ and $W = 13$. We record our iterations in the following table, where each entry will be $(s(j,k))x(j,k)$:

$k \backslash j$	0	$w_1 = 1$	1	$w_1 = 1$	2	$w_1 = 1$	3	$w_1 = 1$	4	$w_1 = 1$	5
0	0		(0)0		(0)0		(0)0		(0)0		(0)0
1	∞		$(0)\infty$		$(0)\infty$		(1)3		(0)3		(0)3
2	∞		$(0)\infty$		(1)2		(0)2		(0)2		(0)2
3	∞		(1)1		(0)1		(0)1		(0)1		(0)1
4	∞		$(0)\infty$		$(0)\infty$		(1)4		(0)4		(0)4
5	∞		$(0)\infty$		(1)3		(0)3		(0)3		(0)3
6	∞		$(0)\infty$		$(0)\infty$		(1)6		(0)6		(0)6
7	∞		$(0)\infty$		$(0)\infty$		$(0)\infty$		(1)10		(0)10
8	∞		$(0)\infty$		$(0)\infty$		$(0)\infty$		$(0)\infty$		$(0)\infty$

Now if we apply the final step of the algorithm, we get $k = 7$, $S = \emptyset$;

$$\begin{aligned} j &= 5 \quad k = 7 \quad S = \emptyset \\ j &= 4 \quad k = 6 \quad S = \{4\} \\ j &= 3 \quad k = 5 \quad S = \{4,3\} \\ j &= 2 \quad k = 3 \quad S = \{4,3,2\} \\ j &= 1 \quad k = 0 \quad S = \{4,3,2,1\}. \end{aligned}$$

So the optimal set is x_1, x_2, x_3, x_4, with payoff $3 + 2 + 1 + 1 = 7$. $\square$