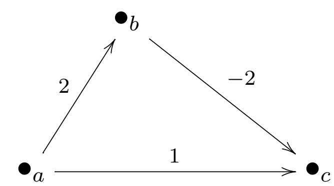
# MATH 409
# WEEK FOUR EXERCISES

**Solution to Exercise 1, Lecture 10.**




Beginning at $a$, we find that $l(c) = 1$ and $l(b) = 2$. The algorithm stops here, despite the fact that the shortest path to from $a$ to $c$ is $abc$ with cost 0. $\square$

**Solution to Exercise 4, Lecture 10.** There are many ways to solve this problem, and we will describe one possibility. We assume that we are given a directed graph $G$ where edges represent streets and vertices represent intersections. Suppose that for some vertex $v_i \in V(G)$, we have some restrictions on the legal paths. Then we do the following: initialize $\overline{G} := G \setminus \{v_i\}$, then

(i) for each edge $e = xv_i \in E(G)$ with target $v_i$ and cost $c_e$, adjoin a new vertex $u_e$ to $\overline{G}$ and add the edge $\alpha_e := xu_e$ to $\overline{G}$ with the same cost $c_e$.
(ii) for each edge $f = v_i y \in E(G)$ with source $v_i$ and cost $c_f$, adjoin a new vertex $w_e$ to $\overline{G}$ and add the edge $\gamma_f := w_e y$ to $\overline{G}$ with the same cost $c_f$.
(iii) if $ef$ represents a legal path in $G$ through $v_i$, then adjoin an edge $\beta_{ef} := u_e w_f$ to $\overline{G}$ with 0 cost.

By construction, every legal path $ef$ in $G$ through $v_i$ corresponds bijectively with a path $\alpha_e \beta_{ef} \gamma_f$ in $\overline{G}$, and the cost of the new path is the same as the old. $\square$

**Solution to Exercise 4, Lecture 12.** The following is a table with the $l(v)$'s values and $p(v)$'s vertices throughout the algorithm. The node added to $R$ in each iteration is in boldface. Observe that once a vertex is added to $R$ its $l$ value does not change, neither does its parent vertex; so once a vertex enters $R$ I will stop writing this values for that vertex on the following iterations.

| v | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) | l(·) | p(·) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | **0** | | | | | | | | | | | | | | | | | |
| a | $\infty$ | | **2** | r | | | | | | | | | | | | | | |
| b | $\infty$ | | 5 | r | **5** | r | | | | | | | | | | | | |
| d | $\infty$ | | $\infty$ | | 10 | a | 10 | a | 10 | a | **9** | k | | | | | | |
| f | $\infty$ | | $\infty$ | | 6 | a | **6** | a | | | | | | | | | | |
| g | $\infty$ | | $\infty$ | | $\infty$ | | $\infty$ | | 9 | f | 9 | f | **9** | f | | | | |
| h | $\infty$ | | $\infty$ | | $\infty$ | | $\infty$ | | $\infty$ | | 16 | k | 14 | d | 13 | g | **13** | g |
| j | $\infty$ | | $\infty$ | | $\infty$ | | $\infty$ | | 13 | f | 13 | f | 13 | f | **12** | g | | |
| k | $\infty$ | | 7 | r | 7 | r | 7 | r | **7** | r | | | | | | | | |

□

**Solution to Exercise 3, Lecture 11.**

| $v$ | Pass 1 | Pass 2 |
|---|---|---|
| $r$ | $(0,\emptyset) \mapsto (0,\emptyset)$ | $(0,\emptyset) \mapsto (0,\emptyset)$ |
| $a$ | $(\infty,\emptyset) \mapsto (2,r)$ | $(2,r) \mapsto (2,r)$ |
| $b$ | $(\infty,\emptyset) \mapsto (5,r)$ | $(5,r) \mapsto (5,r)$ |
| $d$ | $(\infty,\emptyset) \mapsto (10,a) \mapsto (9,k)$ | $(9,k) \mapsto (9,k)$ |
| $f$ | $(\infty,\emptyset) \mapsto (6,a)$ | $(6,a) \mapsto (6,a)$ |
| $g$ | $(\infty,\emptyset) \mapsto (9,f)$ | $(9,f) \mapsto (9,f)$ |
| $h$ | $(\infty,\emptyset) \mapsto (15,d) \mapsto (13,g)$ | $(13,g) \mapsto (13,g)$ |
| $j$ | $(\infty\emptyset) \mapsto (13,f) \mapsto (12,g)$ | $(12,g) \mapsto (12,g)$ |
| $k$ | $(\infty\emptyset) \mapsto (7,r)$ | $(7,r) \mapsto (7,r)$ |

Since there is no change during the second pass, there will not be any changes for the other iterations. So the output is identical to the output at the second pass, and we have found the shortest paths. □

**Solution to Exercise 6, Lecture 12.** Recall that Farkas lemma says that $P \iff Q$, where

$P$ : There exists $x$ such that $Ax \leq b$;
$Q$ : For all $y \geq 0$, $yA = 0 \implies yb \geq 0$.

It is useful to record the negation of these statements:

$\neg P$ : For all $x$, $Ax > b$;
$\neg Q$ : There exists $y \geq 0$, such that $yA = 0$ and $yb < 0$.

Since we know $P \iff Q$, we also know $\neg P \iff \neg Q$. Recall that the statement $P'$ or $Q'$ is equivalent to $\neg P' \implies Q'$. This is what we will prove below.

(i) The statement, there exists $x$ such that $Ax = b$, is equivalent to the statement, there exists $x$ such that $Ax \leq b$ and $Ax \geq b$; this latter condition is the same as

$P'$: there exists $x$ such that

$$\begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix}.$$

We need to prove that $\neg P' \implies \neg Q$. By Farkas' lemmma, $\neg P'$ implies that there exists $y' := (y_1\, y_2) \geq 0$ such that

$$y' \begin{pmatrix} A \\ -A \end{pmatrix} = 0 \qquad \text{and} \qquad y' \begin{pmatrix} b \\ -b \end{pmatrix} < 0.$$

Define $y := y_1 - y_2$, so we have that

$$yA = 0 \qquad \text{and} \qquad yb < 0,$$

which verifies $\neg Q$.

(ii) Let $P'$ be the statement there exists $x \geq 0$ such that $Ax \leq b$. This is equivalent to the statement, there exists $x$ such that

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where $I$ is the identity matrix. By Farkas' lemma, $\neg P' \implies \neg Q'$, so we see that there exists $y := (y_1,\, y_2) \geq 0$ such that

$$(y_1\, y_2) \begin{pmatrix} A \\ -I \end{pmatrix} = 0 \qquad \text{and} \qquad (y_1\, y_2) \begin{pmatrix} b \\ 0 \end{pmatrix} < 0.$$

Equivalently, $y_1 A = y_2 \geq 0$ and $y_1 b < 0$.

(iii) Let $P'$ be the statement there exists $x \geq 0$ such that $Ax = b$. Combining the strategies of the last two parts, we see that this is equivalent to

$$\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix};$$

thus, by Farkas lemma, $\neg P'$ implies that there exists $(y_1\, y_2\, y_3) \geq 0$ such that

$$(y_1\, y_2\, y_3) \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} = 0 \qquad \text{and} \qquad (y_1\, y_2\, y_3) \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix} < 0.$$

Equivalently, if $y := y_1 - y_2$, then $yA = y_3 \geq 0$, and $yb < 0$. $\square$