MATH 409
WEEK 3 EXERCISES

Solution to Exercise 3, Lecture 7. The starting node is v. The list, in order, of the edges added to the tree by Prim's Algorithm is: va, aq, vh, hg, gf, fd, dp, bv. □

Solution to Exercise 4, Lecture 7. As in the proof of correctness for Prim's Algorithm, if T is the subgraph of G found by the proposed algorithm we need to check the three following facts:

(i) T is a spanning subgraph of G.
(ii) T is a tree (i.e. connected and acyclic).
(iii) T is a minimum spanning tree of G.

Proof.

(i) Recall that T is a spanning subgraph of G iff the vertex set of T contains all vertices of G. Beginning with $T = G$, the algorithm does not delete any vertices. Thus, the output T will be a spanning subgraph of G.

(ii) G is assumed connected, and T is initialized at G. At each step of the algorithm, we delete an edge e only if $T \setminus e$ is connected; thus, the final output T must be connected. Assume that the output T is not acyclic; then there is some cycle in T, which contains some edge e of maximal weight (in that cycle). Since e is in a cycle, $T \setminus e$ is connected; this contradicts that the algorithm has terminated.

(iii) We will proceed by contradiction, assume that T is not a minimum spanning tree. Then, by theorem 7(2) of lecture 6, there exists an edge $f = xy \in E(G) \setminus E(T)$ with lower cost than some edge e in the (x, y)-path in T. At some step in the algorithm, we deleted $f = xy$ (since it is in $E(G)$ and not $E(T)$) breaking the cycle formed by f in $T \cup \{e\}$. Observe that e is in this cycle, so that $(T \setminus \{e\}) \cup \{f\}$ is connected. But this contradicts that e has a higher cost than f, since the algorithm deletes edges with higher cost whose deletion does not disconnect the graph. It follows that T is an MST. □

Solution to Exercise 1, Lecture 8. The edges of T, in the order in which Kruskal's algorithm adds them, are: pd, hg, av, df, aq, vh, fg, bv $\square$

Solution to Exercise 4, Lecture 8. By contradiction. Let S be a min-max spanning tree of G and assume that T is an MST but not a min-max spanning tree. If so, there exist an edge $e \in T$ with higher cost than all the edges of S. Let C be a component of $T \setminus e$. Since S is a spanning tree of G then some edge $f \in S$ belongs to $\delta(C)$. For this edge -as well as for all edges of S-we have $c_e > c_f$. This is a contradiction because Theorem 7(3) of lecture 6 states that e must be the edge with lowest cost in $\delta(C)$. $\square$