MATH 409
WEEK ONE EXERCISES

Exercise 7, Lecture 1. P_I is the convex hull of the integer points inside the polyhedron P, i.e. to obtain P_I we **first** have to consider the set of integer points inside P **and then** take the convex hull of that set.

The set $A = \{(x_1, x_2) \in \mathbb{Z}^2 : 2x_1 + x_2 \leq 12, x_2 \leq 11/2, x_1, x_2 \geq 0\}$ is the set of integer points inside P, it is important to notice the difference between A and P_I which is the convex hull of A. How to obtain P_I?. If we draw P and mark the integer points inside it we can see that the vertices of P_I should be $(0, 0), (6, 0), (4, 4), (3, 5)$ and $(0, 5)$ (this is the answer to part (ii)). Using these points we get the inequalities defining P_I to be (sol. to part (i)):

$$P_I = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + x_2 \leq 12, x_2 \leq 5, x_1 + x_2 \leq 8, x_1, x_2 \geq 0\}.$$

(iii) We first observe that the integer points inside P and inside P_I are the same, so in order to solve the integer program of Example 5(b) we can instead solve the integer program $\max\{x_1+x_2 : (x_1, x_2) \in P_I; x_1, x_2$ integer $\}$. But if we just solve the linear program

$$\begin{array}{lll} \text{maximize} & x_1 + x_2 & \\ \text{subject to} & 2x_1 + x_2 & \leq 12 \\ & x_2 & \leq 5 \\ & x_1 + x_2 & \leq 8 \\ & x_1, x_2 \geq & 0 \end{array}$$

we know that the optimum is attained at one vertex (at least), but all the vertices of P_I are integer so the integrality condition will be satisfied automatically.

In this case the two vertices $(4, 4)$ and $(3, 5)$ of P_I are solutions of the integer program above. $\square$

Exercise 11, Lecture 1.

(i) It is clear that there is a one-to-one correspondence between 0-1 incidence vectors (x_1, x_2, x_3) and subgraphs of G. In addition, a subgraph of G will be acyclic if and only if at most one of the two first edges is present, which is the same as saying that for the corresponding incidence vector (x_1, x_2, x_3) only one of the two first variables can be 1. If we only allow the values 0 or 1 for the x_i variables then the equation $x_1 + x_2 \leq 1$ gives exactly that constraint.

Therefore the set of incidence vectors of the acyclic subgraphs of G is

$$A = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : 0 \leq x_1, x_2, x_3 \leq 1, x_1 + x_2 \leq 1\}.$$

Now observe that if (x_1, x_2, x_3) is the incidence vector of a subgraph of G then its weight is exactly $f(x) = 5x_1 + 4x_2 + x_3 =$ sum of the weights of the present edges. Therefore, finding the maximum weight acyclic subgraph of G means to find the incidence vector in A that maximizes $f(x)$.

Observe that P_S is exactly the convex hull of A, so its vertices will belong to A. This implies that the same is true for (at least) one optimal solution of $\max\{5x_1 + 4x_2 + x_3 : 0 \leq x_1, x_2, x_3 \leq 1, x_1 + x_2 \leq 1\}$, and this solution will maximize $f(x)$ in A, so it corresponds to the incidence vector of a maximal weight acyclic subgraph of G.

(ii) Here is an example. The vector (0,0) is the unique feasible solution of the integer program

$$\max\{x_1 + x_2 : 0 \leq x_1, x_2 \leq 0.5; x_1, x_2 \text{ integer }\}.$$

So $x_1 = x_2 = 0$ is the optimal solution of that integer program. But it is easy to see that the optimal solution of the LP-relaxation is $x_1 = x_2 = 0.5$.

(iii) First, we only have integer variables because $(x_1, x_2, x_3) \in \mathbb{Z}^3$. The first three constraints say that the variables can only take values 0 or 1. Now analyze by cases: if $x_3 = 0$ inequality four gives that only one of x_1, x_2 can be 1; if $x_3 = 1$ the fifth constraint also gives that only one of x_1, x_2 can be 1. In summary, the variables can takes values 0 or 1 and the variables x_1 and x_2 cannot be both one at the same time, so the feasible set is the same set A described in (i). By extension

$$A = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1)\}.$$

(iv) The solution to the LP-relaxation is (1, 0.5, 0.5) which is not a solution of the integer problem in (iii) (not feasible because it is not integer).

□

Exercise 12, Lecture 1.

(i) All the vertex packings are:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{3,4\}$$

and their corresponding incidence vectors are

$$S = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,0,1,1)\}.$$

(ii) For part (a) and (b) it is easy to see from the constraint that the variables can only take values 0 or 1. We just need to check that the vectors in S satisfy the equations and that any other vector of 0-1's do not satisfy them. This can be done by inspection.

(iii) The "boundaries" of P are the 3-dim hyperplanes through 4 points of S and which does not split S (just as in $\mathbb{R}^2$ the boundaries of P_I where the lines -1-dim hyperplanes- that goes through 2 points and left all the integer points at one side). By checking all such possible hyperplanes we get that P is described by the linear inequations of (ii)(b), i.e.

$$P = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^2 : x_1+x_2+x_3 \leq 1, x_1+x_2+x_4 \leq 1; x_1, x_2, x_3, x_4 \geq 0\}.$$

Just as a remark, it is not difficult to check that the equations of (ii)(b) are tighter that the ones in (ii)(a): the equations of (ii)(b) readily implies that the variables are between 0 and 1 and the addition of the first and second equation of (ii)(b) gives the first equation of (ii)(a), therefore, a vector satisfying equations (ii)(b) also satisfies the equations (ii)(a). i.e. (ii)(b) is a tighter system than (ii)(a), and strictly tighter because if we drop the integer restriction then the point (0.5, 0, 1, 0) satisfies (ii)(a) but not (ii)(b).

Try to do a similar argument to conclude that the equations of (ii)(b) are tighter than the equations

$$x_1, x_2, x_3, x_4 \geq 0, x_1+x_2 \leq 1, x_1+x_3 \leq 4, x_1+x_4 \leq 1, x_2+x_3 \leq 1, x_2+x_4 \leq 1$$

$\square$

Exercise 1, Lecture 2. We claim that the order is

$$\log(n) < n\log(n) < n^2 < n^{2000} < n^{\log(n)} < 2^{\frac{n}{2}} < 2^n < n! < n^n.$$

We will start at the beginning. From calculus, we know that $\log(n) < n$; it is enough to show $\lim_{n\to\infty} \log(n)/n = 0$ (use L'Hôpital's rule). We have that $\log(n) < n\log(n)$ for $n > 1$; it follows that $n\log(n) < n^2$ for $n \geq 1$. Clearly, $n^2 < (n^2)^{1000} = n^{2000}$ whenever $n^2 > 1$.

Now we move to the end. For the last two inequalities,

$$\begin{array}{rcccccccl} n^n & = & n & n & n & \ldots & n & n & n \qquad n \text{ terms)} \\ n! & = & n & n-1 & n-2 & \ldots & 3 & 2 & 1 \qquad n \text{ terms)} \\ 2^n & = & 2 & 2 & 2 & \ldots & 2 & 2 & 2 \qquad n \text{ terms)} \end{array}$$

so that $2^n < n! < n^n$ for $n > 3$. Since $n/2 < n$ for $n > 0$, and 2^x is an increasing function, we have that $2^{n/2} < 2^n$.

It remains to show that $n^{2000} < n^{\log(n)} < 2^{\frac{n}{2}}$. If we choose $n > e^{2000}$, then

$$n^{2000} = n^{\log(e^{2000})} < n^{\log(n)}.$$

The second inequality, $n^{\log(n)} < 2^{\frac{n}{2}}$, is more delicate. Taking logs, we want to show $\log(n)^2 < \frac{n}{2}\log(2)$ for n sufficiently large. It is easiest to take the limit of the ratio and apply L'Hôpital's rule. Indeed,

$$\lim_{n\to\infty} \frac{\log(n)^2}{\frac{\log(2)}{2} n} = \lim_{n\to\infty} \frac{2\log(n)\cdot\frac{1}{n}}{\frac{\log(2)}{2}} = 0. \qquad \square$$

Solution: Exercise 5, Lecture 3.

(i) $\int \log(x)\,dx = x\log(x) - x + C$, for some $C \in \mathbb{R}$. [This formula holds for $\log(x) = \log_e(x) = \ln(x)$.]

(ii) First, we recall some calc 1. Suppose that f is a strictly increasing differentiable function. Then we know that that any left-endpoint Riemann sum approximation to the integral $\int f(x)\,dx$ is going to be smaller than the integral, and any right-endpoint Riemann sum approximation will be greater than the integral. Now, $f(x) = \log(x)$ is a strictly increasing differentiable function for $x > 0$. Thus,

$$\sum_{k=1}^{n} \log(k) \leq \int_1^{n+1} \log(x)\,dx$$

$$\int_1^{n} \log(x)\,dx \leq \sum_{k=2}^{n} \log(k).$$

Of course, $\log(1) = 0$, so we can combine these inequalities and get

$$\int_1^{n} \log(x) \leq \sum_{k=1}^{n} \log(k) = \log(n!) \leq \int_1^{n+1} \log(x)\,dx.$$

Integrating using part (a), we have

$$n\log(n) - n + 1 \leq \log(n!) \leq (n+1)\log(n+1) - n.$$

From the previous problem, we know that $n\log(n) > n$, so that $n\log(n) - n + 1 = \theta(n\log(n))$. Thus it suffices to show that $(n+1)\log(n+1) - n = \theta(n\log(n))$. Just calculate the limit of the quotient by L'Hôpital's rule:

$$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)\log(n+1) - n}{n\log(n)} &= \lim_{n\to\infty} \frac{\log(n+1) + \frac{1}{n+1}}{\log(n) + 1} \\ &= \lim_{n\to\infty} \frac{n}{n+1} = 1.\end{aligned}$$

(iii) We rewrite $n\log(n) - n + 1 = n\log(n/e) + 1$, and $(n+1)\log(n+1) - n = (n+1)\log(\frac{n+1}{e}) + 1$. Then exponentiating the inequality in (b), we obtain

$$\begin{aligned}&\exp(n\log(n/e) + 1) \leq n! \leq \exp((n+1)\log(n/e) + 1) \\ \iff \quad & e\left(\frac{n}{e}\right)^n \leq n! \leq e\left(\frac{n}{e}\right)^{n+1}.\end{aligned}$$

(iv) We can rewrite that $n! = \Omega(\left(\frac{n}{e}\right)^n)$, and $n! = O(\left(\frac{n}{e}\right)^{n+1})$.

(v) Notice that $\left(\frac{n}{e}\right)^{n+1} > \left(\frac{n}{e}\right)^n$, i.e. these functions are not asymptotic. Stirling's formula gives us a nice function in between these two approximations, which (one can show) more accurately approximates $n!$. Indeed, $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \theta(\left(\frac{n}{e}\right)^{n+.5})$, which is clearly between the previous bounds. $\square$