# Math 409 Midterm Exam Spring 2010

**Name:** SOLUTIONS

1. There are **FOUR** questions in all. Answer all questions.

2. There is a blank sheet at the end that you can tear out and use for scratch work. This sheet does not need to be submitted with the test. If you need extra sheets please ask.

3. **READ THE QUESTIONS CAREFULLY.**

4. Show all your work to get full credit.

5. **No notes are allowed during the test.**

| Problem #1 | Problem #2 | Problem #3 | Problem #4 | Total points |
|---|---|---|---|---|
| | | | | |

**Problem 1** (**8 points**) Calculate the complexity of computing the square of the distance between two points $P$ and $Q$ in $\mathbf{Z^n}$ in the bit complexity model. Start by noting the size of the input.

$P = (p_1, p_2, .., p_n) \in \mathbb{Z}^n$ $Q = (q_1, .., q_n) \in \mathbb{Z}^n$

Let M be max bit complexity of a coordinate in P and Q

(1pt) - Size of input: $2nM$ so $O(nM)$

(2pts) - computing $p_i - q_i$ needs $O(M)$ work and results in an integer of size $O(M)$.

(2pt) - computing $(p_i - q_i)^2$ needs $O(M^2)$ work and results in an integer of size $O(M)$

(1pt) - Doing the above operations $n$ times takes $O(nM + n \cdot M^2) = O(nM^2)$ work

(1pt) Adding the squares $n-1$ times takes $O(nM)$ work

(1pt) Total work $O(nM) + O(nM^2) = O(nM^2)$

**Problem 2** (10 points)

(a) Define a min-max spanning tree in an undirected graph.

(3 pts) A min-max spanning tree has minimal maximum edge weight among the spanning trees of a graph.

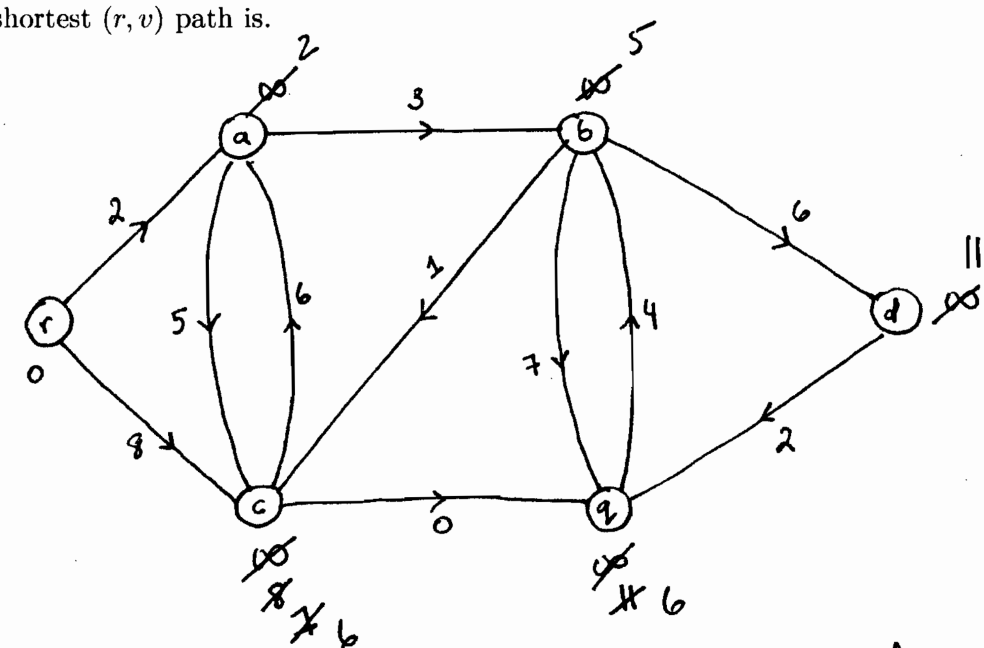
(b) Prove that every minimum spanning tree in an undirected graph is also a min-max spanning tree.

(7 pts) $\boxed{\text{Pf}}$ Assume not. Then there exists an MST, $T$, and a min-max spanning tree $S$ (both spanning trees of our graph) such that the maximum edge weight of $T$ is strictly larger than that of $S$. Let $e$ be an edge in $T$ of greatest edge weight. We know that $T \setminus \{e\}$ consists of two connected components, say $C_1$ & $C_2$. Observe that $e \in \delta(C_1)$; since $S$ is connected, there must be some edge $f \in E(S)$ s.t. $f \in \delta(C_1)$. Observe that $e \neq f$, since $c(e) > c(f)$ by our original assumption. This contradicts that $T$ is an MST: by Theorem 7(3), we would have that $c(e) \leq c(f)$.

In summary, $T$ is not min-max $\Rightarrow$ $T$ is not an MST, i.e.

$T$ is an MST $\Rightarrow$ $T$ is min-max. $\square$

**Problem 3** (**10 points**) (a) Use Dijkstra's algorithm to find a shortest path in the following graph from node $r$ to all other nodes in the graph. For each node $v$ , state clearly what the length of a shortest $(r, v)$ path is.



Sequence in which vertices are chosen

r, a, b, c, q, d

l-values 0 2 5 6 6 11

(b) Write down Dijkstra's algorithm and derive the complexity of its running time.

Set $\ell(r) := 0$ $\ell(v) := \infty$ $\forall\, v \neq r$, $R = \emptyset$

While $R \neq VG$ do

Let $v \in V \setminus R$ s.t $\ell(v) = \min\limits_{w \in V \setminus R} \ell(w)$

$R := R \cup \{v\}$

For all $vw \in E$, $w \notin R$

if $\ell(w) > \ell(v) + c_{vw}$ then

set $\ell(w) := \ell(v) + c_{vw}$

$p(w) := v$

Complexity : $n$ iterations in the while loop

each iteration requires checking & updating all neighbors of the chosen $v$ — $O(n)$ work

Total work: $O(n^2)$.

**Problem 4** (**12 points**) Are the statements below true or false? If false, write down the correction.

(a) Suppose $G$ is the complete graph $K_n$. Then Kruskal's algorithm for finding a spanning tree in this graph runs in $O(n \log n)$ time.

False. Kruskal runs in $O(m \log(n))$. For $G = K_n$, $m = \binom{n}{2} \sim n^2$.

(1 pt) So we have $O(n^2 \log(n))$.

(1 pt)

(b) The simplex algorithm for linear programming runs in polynomial time in the size of the input.

False. For certain classes of programs the run time is exponential.

(1 pt) (1 pt)

(c) Dijkstra's algorithm can be used to find shortest paths in a digraph only if all edge costs are nonnegative while the Moore-Bellman-Ford algorithm can be used with arbitrary edge costs.

False. MBF requires that $G$ contains no negative cycles.

(1 pt) (1 pt)

(d) Farkas lemma states that either $Ax \leq b$ has a solution or there exists $y \leq 0$, $yA \leq 0$ such that $yb < 0$.

False. Farkas' Lemma says, $\exists x$ s.t. $Ax \leq b$ or

(1 pt) $\exists\, y \geq 0$, $yA = 0$ s.t. $yb < 0$.

(1 pt)

(e) If there is a negative cost cycle in a digraph with edge costs, then the graph has no feasible potential but the converse may be false.

False. The converse is true.

(1 pt) (1 pt)

(f) The function $n^{\log n}$ grows faster than the function $n!$.

False. $n!$ grows faster than $n^{\log(n)}$.

(1 pt) (1 pt)