MATH 409 LECTURE 23
MINIMUM COST NETWORK FLOWS

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The minimum cost network flow problem has the following inputs:

- a directed graph $G = (V, E)$,
- edge capacities $u_{ij} \geq 0 \ \forall \ \{i, j\} \in E$,
- demands $b_i \in \mathbf{R} \ \forall \ i \in V$, and
- unit flow costs $c_{ij} \ \forall \ \{i, j\} \in E$.

The minimum cost network flow problem (MCFP) on this data is the problem of finding a feasible flow in G of minimum cost. By a feasible flow we mean a nonnegative flow that does not exceed the edge capacity on every edge. If the flow has to be integral then we have the integral MCFP. This lecture is taken from [1].

There are combinatorial algorithms to solve (MCFP) but in this lecture our aim will be to use the results on TU matrices to solve this problem. Let's start by making a LP formulation of this problem. Using x_{ij} to denote the flow in edge $\{i, j\}$ we have the following LP:

$$\begin{array}{lll} \min & \sum_{\{i,j\}\in E} c_{ij}x_{ij} & \\ \text{s.t.} & -\sum_{\{i,k\}\in E} x_{ik} + \sum_{\{k,i\}\in E} x_{ki} = b_i & \forall i \in V \\ & x_{ij} \leq u_{ij} & \forall \{i,j\} \in E \\ & x_{ij} \geq 0 & \forall \{i,j\} \in E \end{array}$$

The equality constraints say that for each vertex $i \in V$ we need the total inflow minus the total outflow to equal the "demand" b_i. If $b_i > 0$ then b_i can be interpreted as the demand at vertex i (the amount of material that needs to be stored at vertex i), while if $b_i < 0$ then outflow is more than inflow and b_i is like the supply from vertex i. The word "demand" is used in general to stand for our usual notion of supply and demand.

Notice that if we write the constraints of MCFP in matrix notation, then the problem is just

$$\begin{array}{ll} \min & \sum cx \\ \text{s.t.} & M_G x = b \\ & Ix \leq u \\ & x \geq 0 \end{array}$$

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where M_G is the incidence matrix of the digraph G. Rewriting again so that all constraints are inequalities, we have:

$$\begin{array}{ll} \min & \sum cx \\ \text{s.t.} & M_G x \leq b \\ & -M_G x \leq -b \\ & Ix \leq u \\ & x \geq 0 \end{array}$$

This shows that the feasible region of MCFP is of the form

$$P = \{x \in \mathbf{R}^E \,:\, Ax \leq d,\, x \geq 0\}$$

where

$$A = \begin{pmatrix} M_G \\ -M_G \\ I \end{pmatrix}, \quad \text{and} \quad d = \begin{pmatrix} b \\ -b \\ u \end{pmatrix}.$$

Since M_G is TU, the matrix A above is also TU and by using Theorem 7 in Lecture 22 we will have that $P = P^I$ when d is integral. Therefore, the integral MCFP can be solved by ignoring the integrality constraints on the flow variables. The LP-relaxation has an integer optimum and so is also the optimal solution of the integer program for all cost vectors c.

Exercise 1. [1, Problem 4] Consider a scheduling problem in which a machine can be switched on at most k times. Assume that we can break the total duration of time into time periods indexed by t and let $y_t = 1$ if the machine is on in period t and $z_t = 1$ if the machine is switched on in period t.

(1) Argue that the following constraints model the above scheduling problem:

$$\begin{array}{c} \sum_t z_t \leq k \\ z_t - y_t + y_{t-1} \geq 0 \ \ \forall \ t \\ z_t \leq y_t \ \ \forall \ t \\ 0 \leq y_t, z_t \leq 1 \ \ \forall \ t \end{array}$$

(2) Show that the constraint matrix from above is TU.

We now recall some basic facts about linear programs to argue that if A is TU and b is integral, then the linear program

$$\max\, cx \,:\, Ax \leq b,\ x \geq 0$$

has an integer optimum no matter what c is. This will prove one direction of Theorem 7 in Lecture 22.

Adding slack variables s we first transform the above LP to the form

$$\max\, cx + 0s \,:\, Ax + Is = b,\ x, s \geq 0.$$

Now letting $\tilde{c} := (c, 0)$, $\tilde{A} = (A\ I)$ and $\tilde{x} = (x, s)$, we have the LP:

$$\max\ \tilde{c}\tilde{x}\ :\ \tilde{A}\tilde{x} = b,\ \tilde{x} \geq 0$$

At every step of the simplex method we have a dictionary or tableau that records the current basic feasible solution. This involves partitioning the variables $\tilde{x}$ into the basic variables $\tilde{x}_B$ and the nonbasic variables $\tilde{x}_N$ where $\tilde{A} = (B\ N)$ is the partition of the columns of $\tilde{A}$ into the basis B and nonbasis N. The cost vector $\tilde{c}$ also partitions into $\tilde{c} = (\tilde{c}_B, \tilde{c}_N)$ and the problem becomes:

$$\begin{array}{ll} \max & \tilde{c}_B\tilde{x}_B + \tilde{c}_N\tilde{x}_N \\ \text{s.t.} & B\tilde{x}_B + N\tilde{x}_N = b \\ & \tilde{x}_B, \tilde{x}_N \geq 0 \end{array}$$

The basis B is always invertible which means that the problem can be rewritten as:

$$\begin{array}{ll} \max & \tilde{c}_B\tilde{x}_B + \tilde{c}_N\tilde{x}_N \\ \text{s.t.} & \tilde{x}_B = B^{-1}b - B^{-1}N\tilde{x}_N \\ & \tilde{x}_B, \tilde{x}_N \geq 0 \end{array}$$

Now substituting for $\tilde{x}_B$ in the cost function we have the formulation:

$$\begin{array}{ll} \max & \tilde{c}_B B^{-1}b + (\tilde{c}_N - \tilde{c}_B B^{-1}N)\tilde{x}_N \\ \text{s.t.} & \tilde{x}_B = B^{-1}b - B^{-1}N\tilde{x}_N \\ & \tilde{x}_B, \tilde{x}_N \geq 0 \end{array}$$

The basic feasible solution to the simplex method at this stage is: $\tilde{x}_B = B^{-1}b$ and $\tilde{x}_N = 0$. In particular this is how the optimal basic feasible solution looks.

Now if A was TU then $\tilde{A} = (A\ I)$ is also TU. The basis matrix B is an invertible square submatrix of $\tilde{A}$ which means that its determinant is ± 1. This implies that B^{-1} is an integral matrix which in turn implies that the optimal solution $\tilde{x}_B = B^{-1}b$ and $\tilde{x}_N = 0$ is integral for all integral b.

Now recall that we can read off the optimal solution to the dual LP from the final dictionary of the primal LP. The dual LP is

$$\min\ by\ :\ yA \geq c\ y \geq 0$$

and its optimal solution $y^* := \tilde{c}_B B^{-1}N$ where B is basis in the final dictionary of the primal LP. Again, note that this y^* is integral if A is TU. Therefore, we arrive at the following result.

Proposition 2. *If A is TU and b is integral both the LP*

$$\max\ cx\ :\ Ax \leq b,\ x \geq 0$$

and its dual LP

$$\min\ by\ :\ yA \geq c\ y \geq 0$$

have integral optimal solutions.

References

[1] L. Wolsey. *Integer Programming*. Wiley-Interscience, 1998.