MATH 409 LECTURE 22
TOTAL UNIMODULARITY

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Recall that the general integer program is the optimization problem:

$$\max\{cx : Ax \leq b,\, x \in \mathbf{Z}^n\}$$

where we may assume that $c \in \mathbf{Z}^n$, $A \in \mathbf{Z}^{m\times n}$ and $b \in \mathbf{Z}^m$. The **linear relaxation** of this integer program is the problem:

$$\max\{cx : Ax \leq b\}.$$

Let $P := \{x \in \mathbf{R}^n : Ax \leq b\}$ be the feasible region of the linear relaxation. Then recall that P is called a polyhedron and if it is bounded, it is called a polytope. The **integer hull** of P is the convex hull of all the integer points in P and we will denote it as

$$P^I := \text{ conv } \{x \in \mathbf{Z}^n : Ax \leq b\}.$$

If $P = P^I$ then clearly we can solve the integer program given above with respect to any cost vector c by simply solving the linear relaxation. Therefore, a very good question to ask is whether we can tell up front that $P = P^I$ for an integer program. In this lecture we will see a result that answers this question in a stronger way than we are asking. This material is taken from [1].

Definition 1. A matrix A is **totally unimodular** (TU) if every square submatrix of A has determinant 0,1 or -1.

Note that A itself does not have to be square. We just need all the square submatrices of A to have $0, \pm 1$ determinant. However, if A is totally unimodular, then every entry of A has to be $0, \pm 1$ since every entry is a 1×1 submatrix of A.

Exercise 2. Show that A is TU $\Leftrightarrow$ A^T is TU $\Leftrightarrow$ (A, I) is TU $\Leftrightarrow$ $(A, -A, I)$ is TU.

We now prove a result that allows us to find families of TU matrices.

Proposition 3. *A matrix A is TU if the following conditions hold:*

(1) $a_{i,j} = 0, \pm 1$ *for all* i, j.
(2) *each column of A contains at most two nonzero entries.*

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(3) *there is a partition* (M_1, M_2) *of the set* M *of rows of* A *such that for each column* j *of* A *containing two nonzero entries,* $\sum_{i \in M_1} a_{i,j} = \sum_{i \in M_2} a_{i,j}$.

Proof. We will prove this proposition by the method of contradiction. Suppose conditions (1)-(3) hold and A is not TU. Then pick a smallest square submatrix B of A such that $det(B) \neq 0, \pm 1$. Then first note that no column of B is entirely zero since then $det(B) = 0$. Further, no column of B has exactly one nonzero entry since in that case, computing $det(B)$ by expanding along this column shows that there is a smaller submatrix of B whose determinant is not $0, \pm 1$ which contradicts our choice of B. Therefore, every column of B has at least two non-zero entries. On the other hand, condition (2) says that all columns of A have at most two nonzero entries. This means that each column of B has exactly two nonzero entries. Therefore, the nonzero entries in a column of A that that passes through B are in B. Therefore, by condition (3) sum of the rows of B indexed by M_1 equals the sum of the rows of B indexed by M_2 which implies that the rows of B are linearly dependent and so $det(B) = 0$ which is a contradiction. □

Definition 4. (1) The **incidence matrix** M_G of an undirected graph $G = (V, E)$ is the $0/1$ matrix with rows indexed by V and columns by E such that

$$(M_G)_{k,\{i,j\}} = \begin{cases} 0 & \text{if } k \notin \{i,j\} \\ 1 & \text{if } k \in \{i,j\} \end{cases}$$

(2) The **incidence matrix** M_G of a directed graph $G = (V, E)$ is the $0/\pm 1$ matrix with rows indexed by V and columns by E such that

$$(M_G)_{k,\{i,j\}} = \begin{cases} 0 & \text{if } k \notin \{i,j\} \\ -1 & \text{if } k = i \\ 1 & \text{if } k = j \end{cases}$$

The above proposition immediately proves the first of the following results.

Proposition 5. (1) *Incidence matrices of directed graphs are TU.*

(2) *The incidence matrix of an undirected graph* G *is TU if and only if* G *is bipartite.*

Exercise 6. [1, Problem 1] Are the following matrices TU or not?

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

We now come to the connection between TU and the question of when $P = P^I$.

Theorem 7. (1) *For a fixed A, and integral vector b, let $P_b := \{x \in \mathbf{R}^n : Ax \leq b,\ x \geq 0\}$. Then $P_b = P_b^I$ for all $b \in \mathbf{Z}^m$ if and only if A is TU.*

(2) *If A is TU and $b \in \mathbf{Z}^m$ then $P := \{x \in \mathbf{R}^n : Ax \leq b\} = P^I$.*

Exercise 8. [1, Problem 2] Prove that the polyhedron

$$P = \{(x_1, \ldots, x_m, y) \geq 0 : y \leq 1, x_i \leq y\ \forall\ i = 1, \ldots, m\}$$

has the property that $P = P^I$.

References

[1] L. Wolsey. *Integer Programming.* Wiley-Interscience, 1998.