MATH 409 LECTURE 18
APPLICATIONS OF MAX FLOW

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The Transportation Problem [1] The transportation problem is the problem of finding the cheapest way to transport goods from a set of factories P to a set of retailers Q. All factories may not be able to ship to all retailers. Factory $p \in P$ can supply a_p units of goods while retailer $q \in Q$ has a demand for b_q units of goods. We ignore shipping costs and consider the simpler problem of finding a feasible shipment of goods so that the total amount of goods sent from a factory p does not exceed its supply a_p and the total amount of goods received by retailer $q \in Q$ is exactly b_q.

Consider the bipartite graph $G = (P \cup Q, E)$ where E consists of all arcs pq between factories $p \in P$ and retailers $q \in Q$ such that p can ship to q. We need to find $x_{pq} : pq \in E$ such that

$$\begin{aligned} \sum(x_{pq} \,:\, q \in Q,\, pq \in E) &\leq a_p \quad \forall p \in P \\ \sum(x_{pq} \,:\, p \in P,\, pq \in E) &= b_q \quad \forall q \in Q \\ x_{pq} &\geq 0, \text{ integral} \quad \forall pq \in E \end{aligned}$$

Let us call this mathematical formulation $(*)$.

We need the total supply to exceed the total demand for this problem to have a feasible solution. I.e., this problem needs

$$\sum(a_p \,:\, p \in P) \geq \sum(b_q \,:\, q \in Q).$$

Convert the above problem to a network flow problem as follows. Let $G' = (V', E')$ where $V' = P \cup Q \cup \{s, t\}$ and E' consists of the following arcs.

- all arcs $pq \in E$ with infinite capacities
- for each $p \in P$, the arcs sp with capacity a_p
- for each $q \in Q$, the arc qt with capacity b_q

Note that the system $(*)$ has a feasible solution if and only if there exists an integral feasible (s,t)-flow in G' of value $\sum(b_q \,:\, q \in Q)$ which is then a max flow in this network. Thus we can decide if $(*)$ has a

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solution by running the max flow algorithm in G' and checking whether the flow has value equal to or less than $\sum(b_q \,:\, q \in Q)$.

We analyze this problem a bit further. The max flow min cut theorem tells us that $(*)$ has a solution if and only if every (s,t)-cut in G' has capacity at least $\sum(b_q \,:\, q \in Q)$. Consider an arbitrary (s,t)-cut in G'. It has the form $\delta'(\{s\} \cup A \cup B)$ where $A \subseteq P$ and $B \subseteq Q$. What is the capacity of such a general cut? If the capacity of the cut is finite, there does not exists an edge $pq \in E$ such that $p \in A$ and $q \in Q \backslash B$. In this case,

$$\text{capacity}(\delta'(\{s\} \cup A \cup B)) = \sum(a_i \,:\, i \in P\backslash A) + \sum(b_j \,:\, j \in B).$$

Else the cut has infinite capacity. Therefore, $(*)$ has a solution if and only if every finite (s,t)-cut in G' has capacity at least $\sum(b_q \,:\, q \in Q)$ which written mathematically says

$$\sum(a_i \,:\, i \in P\backslash A) + \sum(b_j \,:\, j \in B) \geq \sum(b_j \,:\, j \in Q).$$

Cancelling the common terms on both sides we get

$$\sum(a_i \,:\, i \in P\backslash A) \geq \sum(b_j \,:\, j \in Q\backslash B).$$

Thus $(*)$ has a solution if and only if, for all $A \subseteq P$, $B \subseteq Q$ such that $\delta'(\{s\} \cup A \cup B)$ has finite capacity,

$$\sum(a_i \,:\, i \in P\backslash A) \geq \sum(b_j \,:\, j \in Q\backslash B).$$

Suppose there was a node p in $P\backslash A$ that was not adjacent to a node in $Q\backslash B$. Then if we enlarge A to $A' := A \cup \{p\}$, the capacity of $\delta'(\{s\} \cup A' \cap B)$ lowers since the arc sp is now no longer a cut edge and all other outgoing edges in the new cut were also outgoing edges in the old cut. This means that the left-hand-side in the inequality above lowers making the new inequality stronger than the inequality from the old cut.

Therefore, in order to check the inequality, we can always assume that every node in $P\backslash A$ is adjacent to some node in $Q\backslash B$. For a subset of nodes C in an undirected graph, define the **neighborhood** of C to be the node set $N(C) = \{w \,:\, \{v,w\} \in E \text{ for some } v \in C\}$. Therefore, the previous paragraph shows that in order to check the inequality we only need to consider sets $A \subseteq P$ and $B \subseteq Q$ such that $N(Q\backslash B) = P\backslash A$. In the language of neighborhood sets we see that $(*)$ has a solution if and only if $a(N(C)) \geq b(C)$ for all node sets $C \subseteq Q$.

What does this mean in the context of the original problem? If $C \subseteq Q$ then $b(C)$ is the total demand for the retailers in the set C and

$a(N(C))$ is the total supply from the factories that can ship to C. So we require that the total supply to the retailers in C exceeds the total demand of the retailers in $C \subseteq Q$ for our transportation problem to have a feasible shipping assignment.

References

[1] W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver. *Combinatorial Optimization*. Wiley-Interscience Series in Discrete Mathematics, 1998.