MATH 409 LECTURE 17
APPLICATIONS OF MAX FLOW

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In this lecture we see more applications of the max flow problem. This material can be found in [1, Section 3.3].

Maximum Matchings in Bipartite Graphs

Definition 1. A **matching** in a graph $G = (V, E)$ is a subset M of the edge set E of G such that no two edges in M share a vertex of G.

A **bipartite graph** is an undirected graph with two sets of nodes P and Q such that all edges of H connect a vertex in P to a vertex in Q. Let $H = (P \cup Q, E)$ be a bipartite graph with node sets P and Q.

Problem: Find a matching M in G with $|M|$ as large as possible (i.e., a *maximum matching* in G).

Definition 2. A **cover** of a graph G is a set $C \subseteq V$ such that every edge of G is incident to at least one vertex in C.

Suppose M is a matching in G and C a cover in G. Then every edge in M is incident to at least one vertex in C and moreover, since two edges in M do not share any vertices, there are at least as many vertices in C as there are edges in M. Thus we have the basic inequality

$$|M| \leq |C|.$$

Corollary 3. *If M is a matching in G and C a cover in G such that $|M| = |C|$, then M is a maximum matching in G and C a minimum cover in G.*

This corollary is another illustration of a standard feature in optimization, namely, if we can prove that for two quantities X and Y, each in a family, that $X \leq Y$, then it also follows that the max value of X, among all X's in the family, is less than or equal to the min value of Y, among all Y's in its family. We saw this earlier when we argued that given a particular flow f in a network (G, u, s, t) and a particular

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cut $\delta(A)$, the value of f is at most the capacity of $\delta(A)$ which implies that the max value of any flow in G is at most the min capacity of a cut in G.

Exercise 4. Write down a linear programming formulation of the problem of finding the largest cardinality matching in a graph $G = (V, E)$, and the problem of finding a minimum cover in the same graph. Are these two linear programs related?

In an arbitrary graph, it may not be possible to find a matching and cover of the same cardinality. However, for bipartite graphs, the following famous result is true.

Theorem 5. *(König 1931) For a bipartite graph H*

$$\max\{|M| \,:\, M \text{ is a matching}\} \;=\; \min\{|C| \,:\, C \text{ is a cover}\}.$$

We now describe a max flow algorithm to find a max matching and a min cover in H that also proves the above theorem as a byproduct.

Define a network H' as follows. The vertex set of H' is $P \cup Q \cup \{s, t\}$ and the edge set of H' is the set $\{pq : \text{ the undirected edge } pq \text{ is in } E(H)\}$ $\cup\{sp \,:\, p \in P\} \cup \{qt \,:\, q \in Q\}$. The capacities on the edges are assigned as follows:

$$u_{sp} = 1,\; u_{pq} = \infty,\; u_{qt} = 1.$$

Let x be an integral flow in H' of value k. Then note first that x is a 0/1-valued flow if it comes from a max flow algorithm using augmenting paths. This flow corresponds naturally to a matching M of H defined as $pq \in M$ if $x_{pq} = 1$ and $pq \notin M$ if $x_{pq} = 0$. Check that since x is a $0,1$-flow, M is a matching of H of cardinalty k.

Conversely, suppose M is a matching of H of cardinality k. Then we get an integral 0/1-flow $x = (x_{vw} \,:\, vw \in E(H')\}$ in H' of value k as follows. Let

$$v \in P, w \in Q \quad\Rightarrow\quad x_{vw} = \begin{cases} 1 \text{ if } vw \in M \\ 0 \text{ if } vw \notin M \end{cases}$$

$$v = s, w \in P \quad\Rightarrow\quad x_{vw} = \begin{cases} 1 \text{ if there exists an edge of } M \text{ incident to } w \\ 0 \text{ otherwise} \end{cases}$$

$$v \in Q, w = t \quad\Rightarrow\quad x_{vw} = \begin{cases} 1 \text{ if there exists an edge of } M \text{ incident to } v \\ 0 \text{ otherwise} \end{cases}$$

Note that x is a 0/1-valued (s, t)-flow of value k in H'. Therefore, we have shown that there is a bijection between the 0/1-valued (s, t)-flows in H' and the matchings in H. Thus, in order to solve the max matching problem in H we simply find the max flow in H' and convert the flow into a matching as above.

Can we recover a min cover from this algorithm? Consider a minimum cut $\delta'(\{s\} \cup A)$ in H' with $A \subseteq P \cup Q$ computed from the max flow. Recall that the construction of this min cut is by letting A be the set of vertices reachable from s in the final residual graph G_f. In particular, $t \notin A$. The capacity of this min cut is finite since the capacity of this cut equals the value of the max flow which in turn equals the size of the largest matching in H which is finite since H is a finite graph. This implies that there is no edge in H from $A \cap P$ to $Q \backslash A$. (If there was such an edge in H then it would be in the cut. Since it has infinite capacity, the cut would also have infinite capacity which is a contradiction.) This means that all the edges in H' either go from $P \cap A$ to $Q \cap A$ or from $P \backslash A$ to Q. Therefore, every edge of H is incident to a vertex in $C := (P \backslash A) \cup (Q \cap A)$, and so, C is a cover of H.

What is the capacity of the above min cut $\delta'(\{s\} \cup A)$? By our above observation, there are only two types of edges leaving the set $\{s\} \cap A$: edges of the type sp for all $p \in P \backslash A$ (edges leaving s), and edges of type qt for all $q \in Q \cap A$ (edges leaving $Q \cap A$). An edge of either type has capacity one and there is exactly one edge from s to $p \in P$ and from $q \in Q$ to t. Thus the capacity of the min cut $\delta'(\{s\} \cup A)$ is $|P \backslash A| + |Q \cap A| = |C|$ which by the max-flow min-cut theorem is the value of a max flow in H' which in turn is the size of a max matching in H. Therefore, we conclude that C is a min cover of H since its cardinality equals the size of the matching we found.

Exercise 6. [1, Problem 3.16] Find a max matching and a min cover in the following bipartite graph: $V = \{a, \dots, j, A, \dots, J\}$ and $E = \{aA, aC, aD, bA, bB, bH, cA, cC, dC, dD, eC, eD, fB, fE, fG, gE, gF, gJ, hE, hF, hG, hH, hJ, iI, jD, jI\}$.

Exercise 7. [1, Problem 3.19] Prove that in a matrix the maximum number of nonzero entries, no two in the same line (row or column), is equal to the minimum number of lines that include all the nonzero entries.

Exercise 8. [1, Problem 3.21] Prove that in a bipartite graph G with node set $P \cup Q$ there exists a matching of size $|P|$ if and only if for every subset A of P we have $|N(A)| \geq |A|$ where $N(A)$ is the neighborhood of A which is the set of vertices in Q that are connected to A in G.

Exercise 9. [1, Problem 3.23] Prove that a bipartite graph in which every node has degree exactly $k \geq 1$ has a **perfect** matching. Show by an example that this result is false if we replace "degree k" with "degree at least k".

A **perfect matching** in a graph G is a matching that covers all the nodes of G.

References

[1] W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver. *Combinatorial Optimization*. Wiley-Interscience Series in Discrete Mathematics, 1998.