MATH 409 LECTURE 16
APPLICATIONS OF MAX FLOW

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Elimination of Sports Teams [1]

Suppose we have several teams playing against each other in a sports season and we wish to determine at intermediate points in the season whether certain teams have no chance of winning at the end of the season – i.e., whether certain teams can be eliminated from the running for the winner. For instance, here are two tables that both tell you that team B has no chance of winning the series at the end of the season. This is clear in the first table and perhaps less clear in the second table.

Teams	Wins so far	Games left to play
A	33	8
B	28	4

Note that team B can win a max of 32 games at the end of the season but team A already has 33 wins showing that B is eliminated.

Teams	Wins so far	A	B	C	D
A	33	–	1	6	1
B	29	1	–	0	3
C	28	6	0	–	1
D	27	1	3	1	–

We first work out a condition that guarantees that team B will be eliminated. Define the following notation:

- T – the set of teams other than B
- for all $i \in T$, let $w_i :=$ the number of wins that team i has so far
- for all $i, j \in T$, $i \neq j$ let $r_{ij} :=$ the number of games remaining between i and j
- $P := \{\{i,j\} \subseteq T \,:\, i \neq j,\, r_{ij} > 0\}$. I.e., P consists of all pairs of teams in T that have a positive number of games to play against each other.
- $M =$ number of wins for B at the end of the season if B wins all remaining games

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Proposition 1. *Team B is eliminated if there exists $S \subseteq T$ such that*

$$w(S) + \sum(r_{ij} \, : \, i,j \in S, \, \{i,j\} \in P) > M|S|$$

where $w(S) = \sum(w_i \, : \, i \in S)$.

Proof. Note that $\sum(r_{ij} \, : \, i,j \in S, \, \{i,j\} \in P)$ is the total number of games that are played by pairs of teams in S which equals the total number of wins within S since if a pair within S plays r games then at least one team wins each game. Thus the left hand side of the expression is the total number of wins for teams in S at the end of the season. Bringing $|S|$ to the denominator on the left hand side, we will get that M is strictly smaller than the **average** number of wins for teams in S which means that at least one team in S has more than M wins at the end of the season. Since M was the best that team B can do at the end of the season, we conclude that B is eliminated. □

Remark 2. In our examples, check that in Table 1, $S = \{A\}$ works and in Table 2, $S = \{A, C\}$ works. Note that if S has only one element then the sum $\sum(r_{ij} \, : \, i,j \in S, \, \{i,j\} \in P)$ is zero as there are no pairs of teams in S.

If team B is not eliminated at this point, then there exists a set of possible outcomes for the remaining teams so that team B finishes with the most number of wins. Let y_{ij} be the possible number of wins of team i over team j in the remaining games between the two teams. If B is not eliminated then there is a set of y_{ij}s that make the following system $(*)$ feasible. (Why?)

$$\begin{array}{ccc} y_{ij} + y_{ji} & = \; r_{ij} & \forall \, \{i,j\} \in P \\ w_i + \sum(y_{ij} \, : \, j \in T, \, j \neq i) & \leq \; M & \forall \, i \in T \\ y_{ij} \geq 0, \text{ integral} & & \forall \, \{i,j\} \in P \end{array}$$

We now set up a network $G = (V, E)$ as follows. Let $V = T \cup P \cup \{s, t\}$ and the arc set E be as follows:

- for each $i \in T$, $si \in E$ with capacity $M - w_i$
- for each $i, j \in T$ with $\{i,j\} \in P$, the arcs $i\{i,j\}$ and $j\{i,j\}$ are in E with inifinite capacities
- for each $\{i,j\} \in P$, the arc $\{i,j\}t \in E$ with capacity r_{ij}

Proposition 3. *An integral (s,t)-flow in G of value $\sum(r_{ij} \, : \, \{i,j\} \in P)$ corresponds to a feasible solution to $(*)$ and vice versa.*

Proof. Suppose there is an integral (s,t)-flow in G of value $\sum(r_{ij} : \{i,j\} \in P)$. Then if we choose y_{ij} to be the flow value on the arc $i\{i,j\}$ then we get a feasible solution to $(*)$. Check. Conversely, a solution to $(*)$ yields an integral (s,t)-flow in G by assigning flow y_{ij} to arc $i\{i,j\}$ and defining the flows on the remaining arcs incident to s and t to satisfy flow conservation. Check this and that the flow has value $\sum(r_{ij} : \{i,j\} \in P)$. $\square$

This discussion shows that if B is not eliminated then $(*)$ has a solution which in turn implies that there is an integral (s,t)-flow in G of value $\sum(r_{ij} : \{i,j\} \in P)$. Note that such a flow will be a max flow in G by looking at the capacities on the arcs of G incident to t. Moreover, this max flow will determine a set of outcomes for the remaining games (the y_{ij}s) such that team B finishes first. Conversely, if there is a max flow in G of value $\sum(r_{ij} : \{i,j\} \in P)$, then $(*)$ has a solution which shows that B is not eliminated at this point.

Now we show that if B is eliminated then a minimum cut in G yields a set S as in Proposition 1. Let $\delta(R)$ be a min cut in G. By the max flow min cut theorem, its capacity is less than $\sum(r_{ij} : \{i,j\} \in P)$. Let $S = T \backslash R$. We first show that $R = \{s\} \cup (T\backslash S) \cup \{\{i,j\} \in P : i \,\text{or}\, j \notin S\}$. If i or j is not in $S = T\backslash R$ then i or j in R. If further, $\{i,j\} \notin R$, then this means that $\delta(R)$ has an edge of infinite capacity (leaving the node i or j that is in $R \cap T$ to the node $\{i,j\} \notin R$) which is a contradiction. If $\{i,j\} \in R$ but i and j are not in R (which means they are both in S), then deleting $\{i,j\}$ from R decreases the capacity of the cut by r_{ij} which contradicts that $\delta(R)$ is a min cut. Therefore, R has the above description. Now capacity$(\delta(R))$ =

$$M|S| - w(S) + \sum(r_{ij} : \{i,j\} \in P, \{i,j\} \not\subseteq S) < \sum(r_{ij} : \{i,j\} \in P).$$

Rewriting this inequality we get

$$(w(S) + \sum(r_{ij} : \{i,j\} \in P)) > M|S| + \sum(r_{ij} : \{i,j\} \in P, \{i,j\} \not\subseteq S)$$

or equivalently,

$$w(S) + \sum(r_{ij} : \{i,j\} \in P, \{i,j\} \subseteq S) > M|S|$$

which is the condition from Proposition 1. This shows that B is eliminated and the set $S = T\backslash R$ is the witness.

Exercise 4. [1, 3.31] Suppose we want to know whether it is possible for Team B to finish first or second, that is, so that at most one team has more total wins. How can this be done?

Exercise 5. In this exercise we look at another application of the max-flow min-cut theorem called **optimal closure in a digraph**.

Suppose V is a list of projects that have the following conditions. Each project $v \in V$ has a benefit $b_v \in \mathbb{R}$ and there are certain "closure" restrictions on the projects: if project v is done then project w must also be done. Suppose our goal is to choose a set of projects from the list V such that all closure conditions are satisfied and the total benefit is maximized. There is no limit on the number of projects chosen.

We solve this problem by modeling it as a max flow problem. Construct a digraph $G = (V, E)$ where V is the list of projects and directed edges in E model the closure conditions: if $vw \in E$ then project w has to be chosen if project v is chosen. Then our problem is to choose a max benefit $A \subseteq V$ such that the cut $\delta(A) = \emptyset$. Such a set A is called a closure of V.

A classical example of this problem is in the design of an open-pit mine. Here the region under consideration is divided into three dimensional blocks. For each block v there is an estimated net profit b_v associated with excavating it. If block v is to be excavated, then we must first excavate the block w "above" it. This is modeled by the arc vw which says that if v is chosen to be excavated, we must also excavate w.

Note that we have two extreme situations: if $b_v \geq 0$ for all $v \in V$ then we would choose $A = V$. If $b_v \leq 0$ for all $v \in V$ then we would choose $A = \emptyset$.

Given the digraph G with numbers b_v for all $v \in V$, we create a network G' as follows. The vertex set of G' is $V' = V \cup \{s, t\}$ and the edge set of G' is E' defined as follows. For all $v \in V$ such that $b_v > 0$ put $sv \in E'$ with $u_{sv} = b_v$. For all $v \in V$ such that $b_v < 0$ put $vt \in E'$ with $u_{vt} = -b_v$. Further add $E(G)$ to E' with infinite capacities.

(a) Let $R = \{s\} \cup A$ such that $\delta'(R)$ is a finite capacity cut of G'. Then argue that A is a closure of V. In particular, a min cut works.

(b) If $A \subseteq V$ is a closure of V then show that the (s,t)-cut $\delta'(R)$ has capacity

$$\sum \{b_v \,:\, v \notin A,\, b_v > 0\} - \sum \{b_v \,:\, v \in A,\, b_v < 0\}.$$

(c) Prove that to find a max benefit closure, it suffices to find a min cut $\delta'(\{s\} \cup A)$ of G'. Hint: Adding $\sum\{b_v \,:\, v \in A,\, b_v \geq 0\}$ to both sides of the above expression changes the expression to

$$\sum \{b_v \,:\, v \in V,\, b_v \geq 0\} - b(A).$$

(d) [1, 3.28] Solve the optimal closure problem in the following example. There are 8 jobs to be done with the following names and benefits:

Jobs:	a	b	c	d	e	f	g	h
Benefits:	4	2	-7	3	-1	-3	2	-1

The closure relations are:
If b is done then c has to be done.
If a is done then b and d have to be done.
If d is done then e and g have to be done.
Jobs e and g both need job f to be done.
If h is done then g has to be done.

References

[1] W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver. *Combinatorial Optimization*. Wiley-Interscience Series in Discrete Mathematics, 1998.