MATH 409 LECTURE 12
DETECTING NEGATIVE COST CYCLES IN A DIGRAPH

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The algorithms we saw so far compute the shortest (r, v)-paths in a digraph G from a start vertex r and v any vertex in G. These algorithms require that G has no negative cost cycles, in which case we say that the vector of costs, c, is conservative. In this lecture we see how one can detect whether G has a negative cost cycle. This material is taken from back Section 7.1 in [1].

Definition 1. Let G be a digraph with costs $c_e \in \mathbf{R}$ for all $e \in E(G)$. Let $\pi : V(G) \to \mathbf{R}$ be a function that assigns a real number $\pi(v)$ to every vertex $v \in V(G)$.

(1) For an edge $\{x, y\} \in E(G)$, define the **reduced cost** of $\{x, y\}$ with respect to π to be $c_{xy} + \pi(x) - \pi(y)$.
(2) If the reduced costs of all edges of G are nonnegative, we say that π is a **feasible potential** on G.

Theorem 2. *The digraph G with vector of costs c has a feasible potential if and only if c is conservative.*

Proof. ($\Rightarrow$): Suppose π is a feasible potential on (G, c). Then for every circuit C in G,

$$\sum_{e \in E(C)} c_e = \sum_{xy \in E(C)} c_{xy} + \pi(x) - \pi(y) \geq 0$$

which implies that c is conservative.

($\Leftarrow$): Suppose c is conservative. Augment G to a graph $\tilde{G}$ by adding a new vertex s and putting in edges sv for all $v \in V(G)$ and assigning zero cost to all these new edges. Now run the Moore-Bellman-Ford algorithm on $\tilde{G}$ with the new cost vector $\tilde{c}$ and start vertex s. The algorithm terminates after $|V(G)|$ steps with numbers $l(v)$ on all vertices $v \in V(G)$. Since $l(v)$ is the length of a shortest (s, v)-path in $\tilde{G}$, we have that $l(v) \leq l(w) + c_{vw}$ for all $vw \in E(G)$ and hence the l-values come from a feasible potential on G. $\square$

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Corollary 3. *Given (G, c), in $O(nm)$ time we can find either a feasible potential on G or a negative cost cycle in G.*

Proof. Given (G, c) created the augmented digraph $\tilde{G}$ with vector of costs $\tilde{c}$ exactly as above. Now run the Moore-Bellman-Ford algorithm on $\tilde{G}$ with the new cost vector $\tilde{c}$ and start vertex s. The algorithm terminates after $|V(G)|$ steps with numbers $l(v)$ on all vertices $v \in V(G)$. If l comes from a feasible potential then we are done.

Otherwise, there is some edge $vw \in E(G)$ such that $l(w) > l(v) + c_{vw}$. Recall that in each iteration of the Moore-Bellman-Ford algorithm we check every edge xy and set $l(y) := l(x) + c_{xy}$ if $l(y) > l(x) + c_{xy}$. So the only way we end up with $l(w) > l(v) + c_{vw}$ is because, $l(v)$ changed in the last iteration of the algorithm. But this means that $l(p(v))$ had to change in the last two iterations of the algorithm which means that $l(p(p(v)))$ changed in the last three iterations and so on. Now note that $l(s) = 0$ throughout the algorithm (s has no previous vertex to modify its l-value) and so s cannot be among the set of vertices: $w, v, p(v), p(p(v)), \cdots$. Since the algorithm has $n = |V(G)|$ iterations, there are $n + 1$ elements in the list $w, v, p(v), p(p(v)), \cdots$ which must mean that some vertex is repeated in this list since only n vertices can take part. Therefore, there is a circuit among $w, v, p(v), p(p(v)), \cdots$. Now recall that in the proof of the Moore-Bellman-Ford algorithm we had a set $F := \{xy : x = p(y)\}$ at each stage of the algorithm. The edges encountered in the sequence $w, v, p(v), p(p(v)), \cdots$ are in $F \cup \{vw\}$. Now using the same proof as in the proof of (a) and (b) in the Moore-Bellman-Ford algorithm from the last lecture, we get that the cost of this circuit has to be negative.

Since the Moore-Bellman-Ford algorithm runs on $O(nm)$-time, we get the running time asserted. $\square$

We now relate Theorem 2 to linear programming duality that you know from Math 407. Recall that every linear program (P) has a dual linear program (D) and we may assume that they are of the form:

$$(P): \quad \max cx \,:\, Ax \le b$$

and

$$(D): \quad \min by \,:\, yA = c, \; y \ge 0.$$

We say that a LP is **feasible** if it has a solution. Recall that the dual of the dual of an LP is just the original LP. The most important theorem about linear programs is the duality theorem which says the following.

Theorem 4. *(**Duality Theorem**) Let (P) and (D) are both feasible, then they both have optimal solutions and the optimal values of these programs are the same. Mathematically, if there exists x such that $Ax \le b$ and $y \ge 0$ such that $yA = 0$ then*

$$\max\{cx : Ax \le b\} = \min\{by : yA = c, \ y \ge 0\}.$$

The duality theorem implies another famous result about linear inequality systems called the **Farkas Lemma**.

Theorem 5. *(**Farkas Lemma**) There exists x such that $Ax \le b$ if and only if for all $y \ge 0$ such that $yA = 0$ we get $yb \ge 0$.*

Proof. ($\Rightarrow$): Suppose there exists x such that $Ax \le b$. Then for all $y \le 0$ such that $yA = 0$ we get

$$0 = (yA)x = y(Ax) \le yb.$$

($\Leftarrow$): Consider the LP $-\min\{\sum w_i \ : \ Ax - Iw \le b, \, w \ge 0\}$. Let's write this LP in the form of the problem (P) in the duality theorem to get

$$\max\left\{(0 \ -1)\begin{pmatrix} x \\ w \end{pmatrix} : \begin{pmatrix} A & -I \\ 0 & -I \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} \le \begin{pmatrix} b \\ 0 \end{pmatrix}\right\}$$

which has the solution $x = 0, w = |b|$ which means $w_i = |b_i|$ for each i.

The dual LP is

$$\min\left\{(b, 0)\begin{pmatrix} y \\ z \end{pmatrix} \ : \ (y, z)\begin{pmatrix} A & -I \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \, y, z \ge 0\right\}$$

which is equal to $\min\{by \ : \ yA = 0, 0 \le y \le 1\}$. This dual LP is also feasible since it has the solution $y = 0$. Therefore, by the duality theorem, both programs have optimal solutions and their optimal values are the same.

If $Ax \le b$ is feasible then the original LP has a solution with $w = 0$ which means the value of the LP is 0. Conversely, if the optimal value of the LP is 0, then there is a solution with $w = 0$ which means $Ax \le b$ is feasible. Therefore, $Ax \le b$ is feasible if and only if the optimal value of the first LP is 0 which by the duality theorem is if and only if the optimal value of the dual LP is 0.

Now suppose for all $y \ge 0$ such that $yA = 0$, we have $yb \ge 0$. Then the min value of the dual LP is 0 (attained by setting $y = 0$) which implies that the optimal value of the original LP is 0 which implies that $Ax \le b$ is feasible. $\square$

An equivalent way of stating the Farkas Lemma is as follows:

Farkas Lemma: Either $Ax \le b$ has a solution or there exists $y \ge 0$ such that $yA = 0$ and $yb < 0$.

Check that the "or" in the above statement is exclusive in the sense that you cannot have both: suppose there was x such that $Ax \leq b$ and $y \geq 0$ such that $yA = 0$ and $yb < 0$. Then we get the contradiction:

$$0 = yAx \leq yb < 0.$$

The second version of Farkas Lemma is an example of an **alternative theorem** in mathematics. It says that either A happens or B happens but not both.

Exercise 6. Using Farkas Lemma, establish the following alternative theorems:

(1) Either $Ax = b$ has a solution or there exists y such that $yA = 0$ and $yb < 0$. (We did this in class.)
(2) Either $Ax \leq b$, $x \geq 0$ has a solution or there exists a y such that $y \geq 0$, $yA \geq 0$ and $yb < 0$.
(3) Either $Ax = b$, $x \geq 0$ has a solution or there exists a y such that $yA \geq 0$ and $yb < 0$.

In each case, argue that the two situations are mutually exclusive.

Exercise 7. Show that Theorem 2 follows from Farkas Lemma applied to an appropriate linear inequality system.

References

[1] B. Korte and J. Vygen. *Combinatorial Optimization*. Springer, Berlin, 2000.