The set of all real numbers will be denoted as $\mathbb{R}$ and the set of all integers by $\mathbb{Z}$. A function $f$ is said to be real valued if its values are real numbers. For instance a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ defines the linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $x \mapsto c \cdot x := c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$. Linear programs always have linear objective functions $f(x) = c \cdot x$ as above. Note that this is a real valued function since $c \cdot x \in \mathbb{R}$.

A polyhedron $P \subseteq \mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ that satisfy a finite set of linear inequalities. Mathematically, $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^n$. A polyhedron can be presented in many different ways such as $P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}$ or $P = \{ x \in \mathbb{R}^n : Ax \geq b \}$. All these formulations are equivalent. A polyhedron is called a polytope if it is bounded, i.e., can be enclosed in a ball of finite radius.

**Definition 1.** A set $Q \subseteq \mathbb{R}^n$ is convex if for any two points $x$ and $y$ in $Q$, the line segment joining them is also in $Q$. Mathematically, for every pair of points $x, y \in Q$, the convex combination $\lambda x + (1 - \lambda)y \in Q$ for every $\lambda$ such that $0 \leq \lambda \leq 1$.

**Definition 2.** A convex combination of a finite set of points $v_1, \ldots, v_t$ in $\mathbb{R}^n$, is any vector of the form $\sum_{i=1}^t \lambda_i v_i$ such that $0 \leq \lambda_i \leq 1$ for all $i = 1, \ldots, t$ and $\sum_{i=1}^t \lambda_i = 1$. The set of all convex combinations of $v_1, \ldots, v_n$ is called the convex hull of $v_1, \ldots, v_n$.

Let $\text{conv}(v_1, \ldots, v_n)$ denote the convex hull of $v_1, \ldots, v_n$.

**Theorem 3.** Every polytope $P$ is the convex hull of a finite number of points. If $P = \text{conv}(v_1, \ldots, v_n)$ but $P \neq \text{conv}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, then we say that $v_i$ is a vertex or extreme point of $P$.

A linear program is the problem of maximizing or minimizing a linear function of the form $\sum_{i=1}^n c_i x_i$ over all $x = (x_1, \ldots, x_n)$ in a
polyhedron $P$. Mathematically, it is the problem
\[
\min/\max \sum_{i=1}^{n} c_i x_i : Ax \leq b
\]
for some matrix $A$ and vector $b$.

**Definition 4.** A **discrete optimization problem** is the problem of optimizing (maximizing or minimizing) a real valued objective function over a **discrete** (as opposed to continuous) set of points.

**Example 5.** (a) *A continuous optimization problem:*

\[
\begin{aligned}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad 2x_1 + x_2 \leq 12 \\
& \quad x_2 \leq 11/2 \\
& \quad x_1, x_2 \geq 0
\end{aligned}
\]

This is an example of a linear program. The **feasible region** of this linear program is the polytope

\[
P = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + x_2 \leq 12, \ x_2 \leq 11/2, \ x_1, x_2 \geq 0\}.
\]

The objective or cost function is the real valued function $f(x_1, x_2) = x_1 + x_2$ and the optimal solution is the point $(13/4, 11/2)$ which is a vertex of the feasible region $P$.

(b) *A discrete optimization problem:*

Now consider the **integer program** associated to the above linear program:

\[
\begin{aligned}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad 2x_1 + x_2 \leq 12 \\
& \quad x_2 \leq 11/2 \\
& \quad x_1, x_2 \geq 0 \quad x_1, x_2 \text{ integer}
\end{aligned}
\]

The feasible region of this optimization problem is the set of all lattice points that lie in the polyhedron $P$ from (a). In particular, the feasible region is finite. This integer program has two optimal solutions: $(4, 4)$ and $(3, 5)$.

**Definition 6.** The convex hull of the integer points in a polyhedron $P \subseteq \mathbb{R}^n$ is called the **integer hull** of $P$ and is typically denoted as $P_I$.

**Exercise 7.** Let $P_I$ be the integer hull of the feasible region $P$ in Example 5 (a). In other words, $P_I$ is the convex hull of the feasible solutions to the integer program in Example 5 (b).

(i) Describe $P_I$ using linear inequalities.

(ii) List the vertices of $P_I$. 
(iii) Describe how the integer program in Example 5 (b) can be solved as a linear program using $P_I$.

Let $X$ be a finite set called the ground set and let $2^X$ denote the power set of $X$ which is the set of all subsets of $X$. Recall that $2^X$ has $2^{\mid X \mid}$ elements. If the feasible region of a discrete optimization problem is a subset of the power set $2^X$ then we have a combinatorial optimization problem.

Example 8. Consider the graph $G$ below with the three edges 1, 2, 3. (Graph drawn in class). Let $X$ be the set containing the three edges of $G$. Therefore, $X = \{1, 2, 3\}$. We also assign weights to the three edges as follows: $w_1 = 5$, $w_2 = 4$ and $w_3 = 1$. Now consider the problem of finding the maximum weight acyclic subgraph of $G$. An acyclic subgraph of $G$ is a subgraph of $G$ with no cycles.

The feasible region of the above problem is the set

$\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$

which a subset of the power set $2^{\{1, 2, 3\}}$. The weight of a subgraph is the sum of the weights of its edges. This problem can be solved by first completely enumerating the set of feasible solutions $\mathcal{S}$ as we have done above and then calculating the weights of the solutions and picking the one with largest weight. Check that the optimal solution is the subgraph $\{1, 3\}$. In larger problems the method of complete enumeration followed by sorting may not be practical.

We now turn the above problem into an algebraic/geometric problem and solve it using linear programming.

Definition 9. If $T \subseteq X$, then the incidence or characteristic vector of $T$ is the 0, 1-vector $v(T)$ of length $\mid X \mid$ defined as follows:

$v(T)_x = \begin{cases} 
1 & \text{if } x \in T \\
0 & \text{if } x \not\in T 
\end{cases}$

Example 8 continued. The incidence vectors of the feasible solutions are:

<table>
<thead>
<tr>
<th>$T$</th>
<th>$v(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$(0, 0, 0)$</td>
</tr>
<tr>
<td>${1}$</td>
<td>$(1, 0, 0)$</td>
</tr>
<tr>
<td>${2}$</td>
<td>$(0, 1, 0)$</td>
</tr>
<tr>
<td>${3}$</td>
<td>$(0, 0, 1)$</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>$(1, 0, 1)$</td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>$(0, 1, 1)$</td>
</tr>
</tbody>
</table>
The convex hull of these six vectors is the polyhedron
\[ P_S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, 1 \geq x_3 \geq 0, x_1 + x_2 \leq 1\}. \]

We can now solve this combinatorial optimization problem by solving the linear program \( \max \{5x_1 + 4x_2 + x_3 : (x_1, x_2, x_3) \in P_S\} \). The optimal solution of this linear program is the vertex \((1, 0, 1)\) of \(P_S\) which is the incidence vector of the acyclic subgraph \(\{1, 3\}\). □

The above example shows how one can solve a combinatorial optimization problem by first enumerating its feasible solutions, the set of which is denoted as \(S\). Then we compute an inequality representation of \(P_S = \text{conv}(S)\). Finally we solve the linear program \(\max \{c \cdot x : x \in P_S\}\). The feasible region of this linear program is \(P_S\).

**Definition 10.** Consider the linear program
\[ \max\{c \cdot x : Ax \leq b\} \quad (I) \]
and the associated integer program
\[ \max\{c \cdot x : Ax \leq b, \ x \text{ integer}\} \quad (II). \]

Then (I) is called the **linear programming (LP) relaxation** of (II).

**Exercise 11.**
(i) Why did the linear program
\[ \max \{5x_1 + 4x_2 + x_3 : (x_1, x_2, x_3) \in P_S\} \]
solve the combinatorial optimization problem in Example 8?
(ii) Construct an integer program in two variables whose LP-relaxation does not have the same optimal solution as the integer program.
(iii) Find the feasible region of the following integer program:
\[
\begin{align*}
\max & \quad 5x_1 + 4x_2 + x_3 \\
\text{s.t.} & \quad 1 \geq x_1 \geq 0 \\
& \quad 1 \geq x_2 \geq 0 \\
& \quad 1 \geq x_3 \geq 0 \\
& \quad x_1 + x_2 - x_3 \leq 1 \\
& \quad x_1 + x_2 + x_3 \leq 2 \\
& \quad (x_1, x_2, x_3) \in \mathbb{Z}^3
\end{align*}
\]
(iv) Solve the LP-relaxation of the integer program in (iii). Does it solve the integer program in (iii)?

The point of the above exercise is to illustrate that a combinatorial optimization problem can be formulated as an integer program \(\max \{c \cdot x : Ax \leq b, \ x \text{ integer}\}\) in many different ways (i.e., by choosing different sets of inequalities \(Ax \leq b\)). However, the different LP-relaxations they yield may have different optima. Some are **tighter**
relaxations than others. The inequalities that cut out the convex hull of the feasible region \( S \) of the combinatorial optimization problem yield the tightest linear programming relaxation of the integer program.

**Exercise 12.** A vertex packing of a graph \( G \) is a set of vertices \( F \) of \( G \) with no edges of \( G \) connecting two vertices in \( F \). A vertex packing is sometimes also called an independent set or a stable set in \( G \).

(i) Find all vertex packings of the graph \( G \) whose vertex set is \( V = \{1, 2, 3, 4\} \) and edge set is \( E = \{\{1, 4\}, \{1, 3\}, \{1, 2\}, \{2, 4\}, \{2, 3\}\} \). Calculate their incidence vectors. Let \( S \) be the set of incidence vectors of the vertex packings of \( G \).

(ii) Verify that \( S \) is the feasible region of both the following systems:
(a) \( 2x_1 + 2x_2 + x_3 + x_4 \leq 2, \ 0 \leq x_1, x_2, x_3, x_4 \leq 1, \ x_i \in \mathbb{Z}, i = 1, \ldots, 4 \)
(b) \( x_1 + x_2 + x_3 \leq 1, \ x_1 + x_2 + x_4 \leq 1, \ x_1, x_2, x_3, x_4 \geq 0, \ x_i \in \mathbb{Z}, i = 1, \ldots, 4. \)

(iii) Can you write down a tighter system of inequalities that cut out a polyhedron \( P \) whose set of lattice points is \( S \)? (Hint: What is the inequality presentation of \( \text{conv}(S) \)?)