MATH 409 LECTURE 1
INTRODUCTION TO DISCRETE OPTIMIZATION

REKHA THOMAS

The set of all real numbers will be denoted as $\mathbb{R}$ and the set of all integers by $\mathbb{Z}$. A function f is said to be real valued if its values are real numbers. For instance a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ defines the linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that $x \mapsto c \cdot x := c_1x_1 + c_2x_2 + \cdots + c_nx_n$. Linear programs always have linear objective functions $f(x) = c \cdot x$ as above. Note that this is a real valued function since $c \cdot x \in \mathbb{R}$.

A **polyhedron** $P \subseteq \mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ that satisfy a finite set of linear inequalities. Mathematically,

$$P = \{x \in \mathbb{R}^n \,:\, Ax \leq b\}$$

for some matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^n$. A polyhedron can be presented in many different ways such as $P = \{x \in \mathbb{R}^n \,:\, Ax = b, x \geq 0\}$ or $P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}$. All these formulations are equivalent. A polyhedron is called a **polytope** if it is bounded, i.e., can be enclosed in a ball of finite radius.

Definition 1. A set $Q \subseteq \mathbb{R}^n$ is **convex** if for any two points x and y in Q, the line segment joining them is also in Q. Mathematically, for every pair of points $x, y \in Q$, the **convex combination** $\lambda x + (1 - \lambda)y \in Q$ for every λ such that $0 \leq \lambda \leq 1$.

Definition 2. A **convex combination** of a finite set of points $v_1, \ldots, v_t$ in $\mathbb{R}^n$, is any vector of the form $\sum_{i=1}^t \lambda_i v_i$ such that $0 \leq \lambda_i \leq 1$ for all $i = 1, \ldots, t$ and $\sum_{i=1}^t \lambda_i = 1$.The set of all convex combinations of $v_1, \ldots, v_n$ is called the **convex hull** of $v_1, \ldots, v_n$.

Let $\mathrm{conv}(v_1, \ldots, v_n)$ denote the convex hull of $v_1, \ldots, v_n$.

Theorem 3. *Every polytope P is the convex hull of a finite number of points. If $P = \mathrm{conv}(v_1, \ldots, v_n)$ but $P \neq \mathrm{conv}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, then we say that v_i is a* **vertex** *or* **extreme point** *of P.*

A **linear program** is the problem of maximizing or minimizing a linear function of the form $\sum_{i=1}^n c_ix_i$ over all $\mathbf{x} = (x_1, \ldots, x_n)$ in a

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polyhedron P. Mathematically, it is the problem

$$\min/max \sum_{i=1}^{n} c_i x_i \ : \ A\mathbf{x} \leq b$$

for some matrix A and vector b.

Definition 4. A **discrete optimization problem** is the problem of optimizing (maximizing or minimizing) a real valued objective function over a *discrete* (as opposed to continuous) set of points.

Example 5. (a) *A continuous optimization problem:*

$$\begin{array}{lll} \text{maximize} & x_1 + x_2 & \\ \text{subject to} & 2x_1 + x_2 & \leq 12 \\ & x_2 & \leq 11/2 \\ & x_1, x_2 \geq & 0 \end{array}$$

This is an example of a linear program. The **feasible region** of this linear program is the polytope

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 2x_1 + x_2 \leq 12, \ x_2 \leq 11/2, \ x_1, x_2 \geq 0\}.$$

The objective or cost function is the real valued function $f(x_1, x_2) = x_1 + x_2$ and the optimal solution is the point $(13/4, 11/2)$ which is a vertex of the feasible region P.

(b) *A discrete optimization problem:*
Now consider the **integer program** associated to the above linear program:

$$\begin{array}{lll} \text{maximize} & x_1 + x_2 & \\ \text{subject to} & 2x_1 + x_2 & \leq 12 \\ & x_2 & \leq 11/2 \\ & x_1, x_2 \geq 0 & x_1, x_2 \text{ integer} \end{array}$$

The feasible region of this optimization problem is the set of all lattice points that lie in the polyhedron P from (a). In particular, the feasible region is finite. This integer program has two optimal solutions: $(4, 4)$ and $(3, 5)$.

Definition 6. The convex hull of the integer points in a polyhedron $P \subseteq \mathbb{R}^n$ is called the **integer hull** of P and is typically denoted as P_I.

Exercise 7. Let P_I be the integer hull of the feasible region P in Example 5 (a). In other words, P_I is the convex hull of the feasible solutions to the integer program in Example 5 (b).
(i) Describe P_I using linear inequalities.
(ii) List the vertices of P_I.

(iii) Describe how the integer program in Example 5 (b) can be solved as a linear program using P_I.

Let X be a finite set called the **ground set** and let 2^X denote the power set of X which is the set of all subsets of X. Recall that 2^X has $2^{|X|}$ elements. If the feasible region of a discrete optimization problem is a subset of the power set 2^X then we have a **combinatorial optimization problem**.

Example 8. Consider the graph G below with the three edges $1, 2, 3$. (Graph drawn in class). Let X be the set containing the three edges of G. Therefore, $X = \{1, 2, 3\}$. We also assign weights to the three edges as follows: $w_1 = 5$, $w_2 = 4$ and $w_3 = 1$. Now consider the problem of finding the maximum weight **acyclic** subgraph of G. An acyclic subgraph of G is a subgraph of G with no cycles.

The feasible region of the above problem is the set

$$\mathcal{S} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$$

which a subset of the power set $2^{\{1,2,3\}}$. The weight of a subgraph is the sum of the weights of its edges. This problem can be solved by first completely enumerating the set of feasible solutions $\mathcal{S}$ as we have done above and then calculating the weights of the solutions and picking the one with largest weight. Check that the optimal solution is the subgraph $\{1, 3\}$. In larger problems the method of complete enumeration followed by sorting may not be practical.

We now turn the above problem into an algebraic/geometric problem and solve it using linear programming.

Definition 9. If $T \subseteq X$, then the **incidence** or **characteristic** vector of T is the $0,1$-vector $v(T)$ of length $|X|$ defined as follows:

$$v(T)_x = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}$$

Example 8 continued. The incidence vectors of the feasible solutions are:

T	$v(T)$
$\emptyset$	$(0, 0, 0)$
$\{1\}$	$(1, 0, 0)$
$\{2\}$	$(0, 1, 0)$
$\{3\}$	$(0, 0, 1)$
$\{1, 3\}$	$(1, 0, 1)$
$\{2, 3\}$	$(0, 1, 1)$

The convex hull of these six vectors is the polyhedron

$$P_{\mathcal{S}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, x_1 \geq 0, x_2 \geq 0, 1 \geq x_3 \geq 0, x_1 + x_2 \leq 1\}.$$

We can now solve this combinatorial optimization problem by solving the linear program max $\{5x_1 + 4x_2 + x_3 \; : \; (x_1, x_2, x_3) \in P_{\mathcal{S}}\}$. The optimal solution of this linear program is the vertex $(1, 0, 1)$ of $P_{\mathcal{S}}$ which is the incidence vector of the acyclic subgraph $\{1, 3\}$. $\square$

The above example shows how one can solve a combinatorial optimization problem by first enumerating its feasible solutions, the set of which is denoted as $\mathcal{S}$. Then we compute an inequality representation of $P_{\mathcal{S}} = conv(\mathcal{S})$. Finally we solve the linear program $\max\{c \cdot x \; : \; x \in P_{\mathcal{S}}\}$. The feasible region of this linear program is $P_{\mathcal{S}}$.

Definition 10. Consider the linear program

$$\max\{c \cdot x \; : \; Ax \leq b\} \quad (I)$$

and the associated integer program

$$\max\{c \cdot x \; : \; Ax \leq b, \; x \text{ integer}\} \quad (II).$$

Then (I) is called the **linear programming (LP) relaxation** of (II).

Exercise 11. (i) Why did the linear program

$$\max \; \{5x_1 + 4x_2 + x_3 \; : \; (x_1, x_2, x_3) \in P_{\mathcal{S}}\}$$

solve the combinatorial optimization problem in Example 8?
(ii) Construct an integer program in two variables whose LP-relaxation does not have the same optimal solution as the integer program.
(iii) Find the feasible region of the following integer program:

$$\begin{array}{ll} \max & 5x_1 + 4x_2 + x_3 \\ \text{s.t.} & 1 \geq x_1 \geq 0 \\ & 1 \geq x_2 \geq 0 \\ & 1 \geq x_3 \geq 0 \\ & x_1 + x_2 - x_3 \leq 1 \\ & x_1 + x_2 + x_3 \leq 2 \\ & (x_1, x_2, x_3) \in \mathbb{Z}^3 \end{array}$$

(iv) Solve the LP-relaxation of the integer program in (iii). Does it solve the integer program in (iii)?

The point of the above exercise is to illustrate that a combinatorial optimization problem can be formulated as an integer program $\max\{c \cdot x \; : \; Ax \leq b, \; x \text{ integer}\}$ in many different ways (i.e., by choosing different sets of inequalities $Ax \leq b$). However, the different LP-relaxations they yield may have different optima. Some are *tighter* relaxations than others. The inequalities that cut out the convex hull of the feasible region $\mathcal{S}$ of the combinatorial optimization problem yield the tightest linear programming relaxation of the integer program.

Exercise 12. A **vertex packing** of a graph G is a set of vertices $\mathcal{F}$ of G with no edges of G connecting two vertices in $\mathcal{F}$. A vertex packing is sometimes also called an **independent set** or a **stable set** in G.

(i) Find all vertex packings of the graph G whose vertex set is $V = \{1,2,3,4\}$ and edge set is $E = \{\{1,4\},\{1,3\},\{1,2\},\{2,4\},\{2,3\}\}$. Calculate their incidence vectors. Let $\mathcal{S}$ be the set of incidence vectors of the vertex packings of G.

(ii) Verify that $\mathcal{S}$ is the feasible region of both the following systems:
(a) $2x_1 + 2x_2 + x_3 + x_4 \leq 2, \; 0 \leq x_1, x_2, x_3, x_4 \leq 1, \; x_i \in \mathbb{Z}\, i = 1, \ldots, 4$
(b) $x_1 + x_2 + x_3 \leq 1, \; x_1 + x_2 + x_4 \leq 1, \; x_1, x_2, x_3, x_4 \geq 0, \; x_i \in \mathbb{Z}, \, i = 1, \ldots, 4$.

(iii) Can you write down a tighter system of inequalities that cut out a polyhedron P whose set of lattice points is $\mathcal{S}$? (Hint: What is the inequality presentation of $\mathrm{conv}(\mathcal{S})$?)