Math 408 Final Exam Winter 2008

Name:

1. There are **FOUR** questions in all. Several questions have multiple parts. Answer all questions.
2. There is a blank sheet at the end that you can tear out and use for scratch work. This sheet does not need to be submitted with the test. If you need extra sheets please ask.
3. **READ THE QUESTIONS CAREFULLY.**
4. Show all your work to get full credit.
5. **No notes or calculators are allowed during the test.**

Problem #1	Problem #2	Problem #3	Problem #4	Total points

Problem 1 (10 points)

(i) Prove that if $g_1, \dots, g_l, h_1, \dots, h_m$ are continuous functions from $\mathbf{R}^n \to \mathbf{R}$, then

$$D = \{\mathbf{x} \in \mathbf{R}^n \, : \, g_i(\mathbf{x}) \leq 0 \;\; i = 1, \dots, l, \;\; h_i(\mathbf{x}) = 0 \;\; i = 1, \dots, m\}$$

is a closed set.

(ii) Rewrite clearly the definition of a closed set used in (i)

(iii) Prove that if $g_1, \ldots, g_l$ are convex functions and $h_1, \ldots, h_m$ are affine functions from $\mathbf{R}^n \to \mathbf{R}$, then the set D in (i) is a convex set.

(iv) Define a convex set in $\mathbf{R}^n$ (as you used it in (iii)).

Problem 2 (**10 points**) In each of the problems below, decide if there is a global optimum by using a result from class, when applicable. If you conclude there is a global optimum explain clearly which result you applied and show all the needed checks. If you are unable to conclude that there is a global optimum using the results from class, then say what fails.

(i)

$$\begin{aligned} \min \quad & \sin(x_1 x_2 + (x_1 - 3)^2) \\ \text{s.t.} \quad & 16x_1^2 + 25x_2^2 \leq 100 \\ & x_1 \geq 1, x_2 \geq 1 \end{aligned}$$

(ii) min $\quad 5x_1^2 + 2x_1x_2 + x_2^2 - x_1 + 2x_2 + 16$

(iii)

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 + \cdots + x_n^2 \\ \text{s.t.} \quad & a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \end{aligned}$$

where $a_1, \ldots, a_n, b \in \mathbf{R}$.

Problem 3 (**10 points**) Find all local optima of the following problem by setting up the KKT condition and solving for all feasible solutions that satisfy KKT. Show all your work.

$$\begin{aligned} \min \quad & e^{x_1} - x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leq 1 \\ & x_2 \geq 0 \end{aligned}$$

Problem 4 (10 points) Are the following statements true or false? Give clear reasons or a counterexample to support your answer.

(i) Every polynomial of even degree has a local minimum.

(ii) If $f : \mathbf{R}^n \to \mathbf{R}$ is a coercive function, then $\phi(t) := f(\mathbf{u} + t\mathbf{w})$ is also coercive, where $\mathbf{u}, \mathbf{w}$ are fixed vectors in $\mathbf{R}^n$ and $t \in \mathbf{R}$.

(iii) Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$. Let γ be any real number such that $\gamma + \lambda_1 > 0$. Then $\gamma I_n + A$ is positive definite.

(iv) The steepest descent direction for the function $f = x_1^2 + x_2^2 + x_1 x_2 - x_1 + 500$ at the point $(1, 1)$ is $(-2, -3)^t$.

SCRATCH PAPER