Math 408 Homework 1

1. Solve the following linear system of equations:

$$\begin{array}{rrrrrrcl} 2x_1 & & & +x_4 & +x_5 & & = & b_1 \\ & 2x_2 & & +x_4 & & +x_6 & = & b_2 \\ & & 2x_3 & & +x_5 & +x_6 & = & b_3 \end{array}$$

where (b_1, b_2, b_3) denotes an arbitrary right hand side vector for the system.

(i) Is there any value of (b_1, b_2, b_3) for which this system is infeasible? If not explain why. If yes, find such a value and justify your answer.

(ii) How many solutions does this system have when it is feasible?

(iii) Write down one solution of this system for the right hand side vector $(10, 10, 10)$. Show your work.

(iv) How does the solution space of this system look?

(v) What is the dimension of the solution space of this system?

(vi) In general, what are the possibilities for the number of solutions of a linear system $A\mathbf{x} = \mathbf{b}$?

2. The general $0, 1$**-integer program** is the following problem:

$$\begin{array}{l} \max \mathbf{cx} \\ \text{s.t. } A\mathbf{x} \leq \mathbf{b} \\ x_i = 0, 1 \text{ for all } i = 1, \ldots, n \end{array}$$

where $\mathbf{c} \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$ and $\mathbf{b} \in \mathbf{R}^m$ form the data in the problem and $\mathbf{x} = (x_1, \ldots, x_n)$ is the vector of variables. Express the general $0, 1$-integer program as a nonlinear program. Justify your answer.

3. Many important problems in optimization can be formulated as a $0, 1$-integer program. One such problem is that of finding the maximum size of a stable set in a grah G which is described as follows:

Let $G = (V, E)$ be an undirected graph where V is the set of vertices and E the set of edges in G. A subset $S \subseteq V$ is called a **stable or independent set** in G if for every pair of vertices in S, there is no edge connecting them in G. The size of a stable set S is the cardinality, $|S|$, of the set S. The maximum stable set problem asks to find the largest stable set in a graph G.

For example, in a pentagon, the maximum size of a stable set is two. The complete graph K_n is the graph on n vertices in which every pair of vertices form an edge. The maximum size of a stable set in K_n is one. Finding the maximum size of a stable set in a graph has important applications in communication systems and coding theory and hence the problem has received a lot of attention in computer science and combinatorial optimization.

Formulate the maximum stable set problem in G as a nonlinear program using linear and quadratic polynomials. Justify your answer completely.

4. Consider the following linear program:

$$\begin{aligned} &\max\ x_1 + x_2 + 3x_3 \\ &\text{s.t. } 0 \leq x_i \leq 1 \quad i = 1, 2, 3 \\ &2x_1 + x_2 + 3x_3 \leq 5 \end{aligned}$$

(i) Draw the set of solutions (the feasible region) of this LP in $\mathbf{R}^3$.

(ii) Find all the extreme points of this feasible region.

(iii) Find an optimal solution of this linear program from parts (i) and (ii). Justify your answer. (You don't need to use the simplex method or anything sophisticated. Just some geometry.)