Math 408 Midterm Exam Winter 2011 Solutions.

Name:

1. There are **THREE** questions in all. Answer all questions.

2. There is a blank sheet at the end that you can tear out and use for scratch work. This sheet does not need to be submitted with the test. If you need extra sheets please ask.

3. **READ THE QUESTIONS CAREFULLY.**

4. Show all your work to get full credit.

Problem #1	Problem #2	Problem #3	Total points

Problem 1 (12 points) Consider the function $f(x, y, z) = (x - y)^2 + e^z - z$. $= x^2 + y^2 - 2xy + e^z - z$

(a) Compute the gradient of f.

$$\nabla f = \begin{pmatrix} 2x - 2y \\ 2y - 2x \\ e^z - 1 \end{pmatrix}$$

(b) Describe (mathematically) the set of points (x, y, z) at which the gradient vanishes.

$\nabla f = 0 \Rightarrow e^z = 1 \Rightarrow z = 0$ & $x = y$

$\therefore$ Pts at which $\nabla f = 0$ is $\{(t, t, 0) : t \in \mathbb{R}\}$

(c) Compute the Hessian of f.

$$Hf = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & e^z \end{pmatrix}$$

(d) Is the Hessian positive definite at any of the points at which the gradient vanishes?

$$Hf(t,t,0) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

principal minors:

1×1: $2, 2, 1$

2×2: $0, 2,$

3×3: $1(4-4) = 0$

leading principal minors: $2, 0,$ $\therefore$ Hf not positive definite at any point at which $\nabla f = 0$

(e) Is the Hessian positive semidefinite at any of the points at which the gradient vanishes?

The Hessian is positive semidefinite at all points at which $\nabla f = 0$

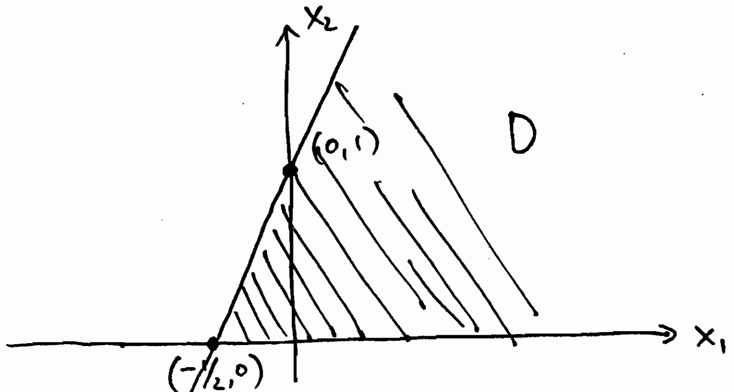
(f) What can you conclude about the local minima of this function?

Cannot conclude if $\exists$ points of the form $(t,t,0)$ that are local minima. since $Hf(t,t,0)$ is not positive definite at any such point.

Cannot also say that one such point is <u>not</u> a local min since $Hf(t,t,0) \succeq 0$.

Problem 2 (13 points) (a) Draw the following set and label your figure carefully.

$$D = \{(x_1, x_2) : -2x_1 + x_2 \le 1,\ x_2 \ge 0\}$$



(b) Does $f = x_1^4 + x_2^4 - x_1^2 x_2^2$ have a global minimum over D? Why?

$D \neq \emptyset$, D is closed since its defining inequalities $g_i(x) \le 0$ have g continuous.

f is coercive since $f = (x_1^2 - x_2^2)^2 + x_1^2 x_2^2$

as $\|x\| \to \infty$ at least one of $x_1^2 \to \infty$ or $x_2^2 \to \infty$ $\Rightarrow x_1^2 x_2^2 \to \infty$ and $(x_1^2 - x_2^2)^2 \ge 0$ always $\therefore f \to \infty$

$\therefore$ f has a global min over D.

(c) Does $f = e^{x_1} - x_2$ have a global minimum over D? If so, find it.

Since $-x_2 \ge -2x_1 - 1 \quad \forall\ (x_1, x_2) \in D$

$e^{x_1} - x_2 \ge e^{x_1} - 2x_1 - 1 \quad \forall\ (x_1, x_2) \in D$

min is achieved on the line $-2x_1 + x_2 = 1$

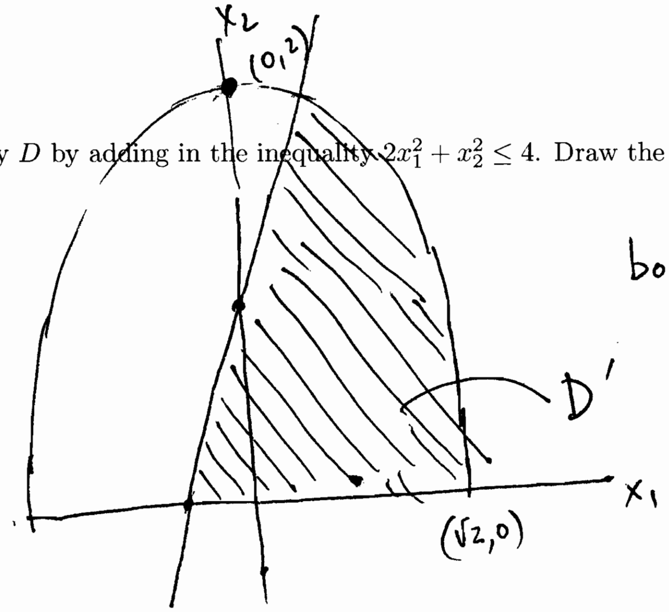
$\therefore$ ~~min value is~~ Plugging into objective function get the 1-variable function $e^{x_1} - 1 - 2x_1$

$\frac{d}{dx} e^{x_1} - 1 - 2x_1 = e^{x_1} - 2 = 0 \Rightarrow e^{x_1} = 2 \Rightarrow x_1 = \ln 2$

$\boxed{\text{Min pt } (\ln 2,\ 1 + 2\ln 2)}$

$\frac{d^2}{dx^2}(e^{x_1} - 1 - 2x_1) = e^{x_1} \ge 0$ when $x_1 = \ln 2$

(d) Modify D by adding in the inequality $2x_1^2 + x_2^2 \leq 4$. Draw the new region.



bounded feasible region

(e) Does $f = x_1^4 + x_2^4 - 3x_1^2x_2^2$ have a global minimum over this new region?

Yes since f is continuous and $D' \neq \phi$, D' closed (since it's defined by inequalities given by continuous functions) and D' bounded. ($D' \subseteq$ ellipse $2x_1^2 + x_2^2 \leq 4$)

(f) Could you also conclude that in (e) f has a global minimum over the old region D?

$$x_1^4 + x_2^4 - 3x_1^2x_2^2 = (x_1^2 - x_2^2)^2 - x_1^2x_2^2$$ when $x_1 = x_2$ and $|x_1| = |x_2| \to \infty$ $f \to -\infty$. $\therefore f$ not coercive.

$\therefore$ We cannot conclude that f has a global min over D.

Problem 3 (15 points) Are the following statements true or false? If false, state the correction precisely or give an example to show that the statement is false.

(a) A closed set is always bounded.

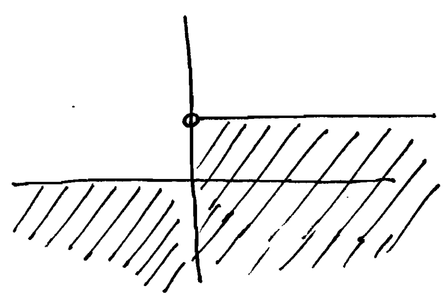
False. The set D in Prob 2 (a) is closed but not bounded.

(b) Every set of the form $\{x : g(x) \leq 0\}$ is closed.

False. If g was continuous the statement is true.

Define $g(x,y) = \begin{cases} y & \text{when } x \leq 0 \\ y-1 & \text{when } x > 0 \end{cases}$

Then $g(x,y) \leq 0$ not closed



(c) Every linear program has a local minimum.

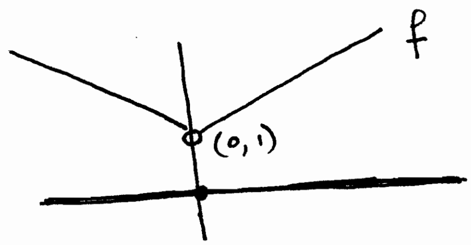
True if the LP is feasible and bounded

False otherwise

(d) A non-linear program in one variable always has at least a local (if not global) minimum.

False. If the problem is infeasible there is no local min

Ex



$\min_{x \in \mathbb{R}} f$ has no local or global min

(e) The Hessian of every function in n variables, for which the second partials exist, has nonnegative eigenvalues.

False. True iff the Hessian is psd