3.2.2 a) $f = x_1^3 + x_2^3 + x_3^3 - x_1 x_2$ f not coercive.

Set $x_1 = x_2 = 0$ and let $x_3 \to -\infty$. Then $\|x\| \to \infty$ but $f \to -\infty$

c) $f = x_1^4 + x_2^4 + x_3^2 - 7 x_1 x_2 x_3^2$ f not coercive
Set $x_1 = x_2 = x_3$ and let $x_1 \to \infty$ then
$f = 2x_1^4 + x_1^2 - 7x_1^4 = -5x_1^4 + x_1^2 \to -\infty$

e) $f = \ln(x_1^2 x_2^2 x_3^2) - x_1 - x_2 - x_3$ f not coercive
$= 2\ln(x_1) + 2\ln(x_2) + 2\ln(x_3) - x_1 - x_2 - x_3$

Set $x_1 = x_2 = 0$ let $x_3 \to \infty$ then $f = 2\ln x_3 - x_3 \to -\infty$

g) $f = x_1^4 + x_2^4 - x_1^2 x_2^2 = (x_1^2 - x_2^2)^2 + x_1^2 x_2^2 \to \infty$

as $\|x\| \to \infty$. f is coercive.

3.2.5 (2.4.2) $\min\ x_1 + x_2^2$
s.t $3x_1 + 2x_2 = 3$
$x_1, x_2 \geq 0$

feasible region is nonempty: $(1,0)$ is a sol$^{\underline{n}}$
closed by Lemma 3.1.3 and bounded: lies in
$B(0,2)$ for instance.
$f = x_1 + x_2^2$ continuous since its a polynomial.
$\therefore$ By Thm 3.2.1 $\exists$ a global min

3.2.8 $\min \sum_{i=1}^{n} x_i^2$

s.t $a_1 x_1 + \cdots + a_n x_n = b$

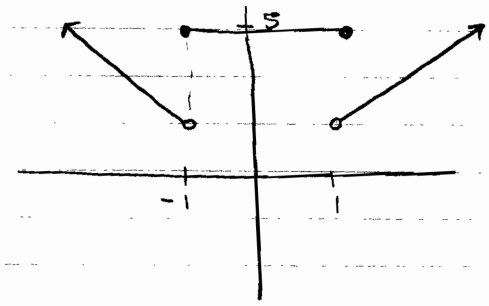
$D = \{x \in \mathbb{R}^n : \sum a_i x_i = b\}$
$D \neq \emptyset$ since $x_1 = b/a_1$, $x_2 = x_3 = \cdots = x_n = 0$ is a sol$^{\underline{n}}$ to $\sum a_i x_i = b$

$f = \sum x_i^2$ coercive since $f = \|x\|^2$ $\therefore \|x\| \to \infty \Rightarrow f \to \infty$

Further all functions involved are continuous $\therefore$ By
Cor 3.2.1 $\exists$ a global min

3.2.9 a) Take the constant function $f(x) = 1$. Then every $x \in \mathbb{R}$ is a global min of this function but $f(x) \not\to \infty$ as $\|x\| \to \infty$.

b) $f(x) = \begin{cases} |x| & \text{when } x > 1 \text{ and } x < 1 \\ 5 & \text{when } -1 \le x \le 1 \end{cases}$



This function $\to \infty$ as $\|x\| \to \infty$ but its global min value is never achieved.

3.2.10 Fix α, R

"$f(x) > \alpha$ whenever $\|x\| > R$" $\iff$ $\|x\| > R \Rightarrow f(x) > \alpha$ — ①

"$\|x\| \le R$ whenever $f(x) \le \alpha$" $\iff$ $f(x) \le \alpha \Rightarrow \|x\| \le R$ — ②

Contrapositive of ① is $f(x) \le \alpha \Rightarrow \|x\| \le R$ which is ②
So ① & ② are equivalent

3.2.3 $f(x) = a^T x \quad a \in \mathbb{R}^n \Rightarrow f$ not coercive.

To see this, if some a_i is < 0 then take all variables except x_i to be zero and send $x_i \to \infty$. Then $f(x) \to -\infty$. If all $a_i \geq 0$ pick an $a_i > 0$ and send $x_i \to -\infty$ keeping all other variables at 0. Then again $f(x) \to -\infty$.

$$g(x) = a^T x + \varepsilon \|x\|^2 = a^T x + \varepsilon (x_1^2 + x_2^2 + \cdots + x_n^2) \qquad \varepsilon > 0$$

$$= (a_1 x_1 + \varepsilon x_1^2) + (a_2 x_2 + \varepsilon x_2^2) + \cdots + (a_n x_n + \varepsilon x_n^2)$$

Each term is dominated by the square term as $|x_i| \to \infty$. and so $g \to \infty$. $\therefore g$ is coercive.

3.2.4
$$\min \; x_1 + x_2^2$$
$$\text{s.t } 3x_1 + 2x_2 = 3$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

Check that the feasible region is bounded, closed and nonempty.

Then since $f = x_1 + x_2^2$ is continuous, by Thm 3.2.1 $\exists$ a global opt.

3.2.5
$$\min \; x_1 - x_2^2$$
$$\text{s.t } 3x_1 + 2x_2 = 3$$
$$x_1 \geq 0, \; x_2 \geq 0$$

Same feasible region and conts f. So by Thm 3.2.1 $\exists$ a global opt.

3.2.6
$$\min \; x_1 x_3 + x_2^2$$
$$\text{s.t } x_1^2 + x_2^2 + x_3^4 = 4$$

f is continuous everywhere

feasible region is $D = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^4 = 4\}$

$(2,0,0) \in D \quad \therefore D \neq \emptyset$

D is closed by Lemma 3.1.3

D is bounded since if $(x_1, x_2, x_3) \in D$ then

$x_1^2 \leq 4$, $x_2^2 \leq 4$ and $x_3^4 \leq 4 \Rightarrow -2 \leq x_1 \leq 2, \; -2 \leq x_2 \leq 2$

and $-2 \le x_3^2 \le 2 \Rightarrow -\sqrt{2} \le x_3 \le \sqrt{2}$.

$\therefore$ By Thm 3.2.1 $\exists$ a global opt.

3.2.7 $\min \quad c_1x_1 + \cdots + c_nx_n$ ← linear f not coercive

$\text{s.t} \quad x_1^2 + x_2^2 + \cdots + x_n^2 \le b$

feas region $D = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \le b\}$ is bounded since it lies in $B(0, \sqrt{b})$

$D \ne \emptyset$ since $(\sqrt{b}, 0, 0, \ldots 0) \in D$ or $(0, 0, \ldots 0) \in D$

D closed by Lemma 3.1.3.

$\therefore$ By Thm 3.2.1 $\exists$ a global opt.