Homework 4 Homework 4

3.1.3 $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$

a) f is a polynomial in x_1, x_2, x_3. So it's continuous everywhere.

b) $\frac{\partial f}{\partial x_1} = 2x_1 + 2x_2 - 4x_3$

$\frac{\partial f}{\partial x_2} = 4x_2 + 2x_1 + 6x_3$

$\frac{\partial f}{\partial x_3} = -6x_3 - 4x_1 + 6x_2$

$$\nabla f = \begin{pmatrix} 2x_1 + 2x_2 - 4x_3 \\ 4x_2 + 2x_1 + 6x_3 \\ -6x_3 - 4x_1 + 6x_2 \end{pmatrix}$$

Partials are defined everywhere, so f is differentiable everywhere

c) $Hf = \begin{pmatrix} 2 & 2 & -4 \\ 2 & 4 & 6 \\ -4 & 6 & -6 \end{pmatrix}$ f twice-differentiable everywhere

d) $f(x) = f(x^*) + \nabla f(x^*)^T(x - x^*) + \frac{1}{2}(x - x^*)^T Hf(z)(x - x^*)$

for some $z \in [x, x^*]$

Letting $x^* = (0,0,0)$ get

$f(x) = f((0,0,0)) + \nabla f((0,0,0))^T x + \frac{1}{2} x^T Hf(z)(x)$

$= 0 + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} x + \frac{1}{2}(x_1, x_2, x_3)^T \begin{bmatrix} 2 & 2 & -4 \\ 2 & 4 & 6 \\ -4 & 6 & -6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ since $Hf(x)$ is a constant matrix at all $x \in \mathbb{R}^3$.

3.1.5 ① $D_1 = \{x \in \mathbb{R} : x \text{ integer}\}$ is closed since its complement in $\mathbb{R}$ is an infinite union of open intervals.

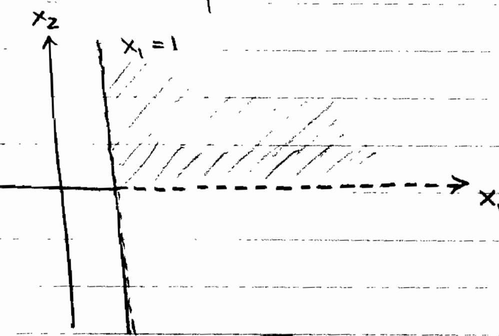
② $D_2 = \{x \in \mathbb{R} : x \text{ rational \# in } [0,1]\}$ not closed since for any irrational # p in $[0,1]$ $\exists$ a sequence of rational numbers in $[0,1]$ converging to p but the limit pt is not in D_2.

$D_3 = \{(x_1, x_2) \in \mathbb{R}^2 : \quad e^{x_1} \le x_2 \; , \; \sin(x_1 + x_2) = 0\}$

By Lemma 3.1.3 this set is closed since all the fcns involved are continuous.

$D_4 = \{x \in \mathbb{R}^3 : \; x_1 \ge 1, \; 1/x_2 \ge 0\}$

Not closed. Consider the sequence $(1, 1/k)$ as $k \to \infty$. This sequence has limit $(1,0)$. But $(1,0)$ not in D_4



$D_5 = \{x \in \mathbb{R}^n : 1 \le \|x\| \le 3\} = \{x \in \mathbb{R}^n : 1 \le \sqrt{x_1^2 + \cdots + x_n^2} \le 3\}$

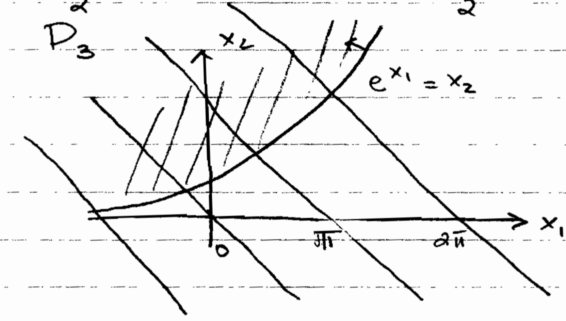
$= \{x \in \mathbb{R}^n : \; 1 \le \sum x_i^2 \le 9\} = \left\{x \in \mathbb{R}^n : \begin{array}{l} \sum x_i^2 \le 9 \\ \sum x_i^2 \ge 1 \end{array}\right\}$

closed by Lemma 3.1.3

3.1.6: D_1 not bounded

D_2 bounded $\quad D_2 \subset B(0, 2)$

D_3



D_3 = intersection of the lines $x_1 + x_2 = n\pi$ with the region $e^{x_1} \le x_2$

Not bounded.

D_4 not bounded

D_5 bounded $\quad D_5 \subseteq B(0, 3)$

3.1.7 $\; f : \mathbb{R}^n \to \mathbb{R}$ continuous fcn. $\quad \alpha$ fixed #

Show $D = \{x \in \mathbb{R}^n : f(x) \ge \alpha\}$ is closed.

Proof: Consider any sequence of vectors $x^{(1)}, x^{(2)}, \ldots$ in D that converges to some $x \in \mathbb{R}^n$.

Then $f(x^{(i)}) \ge \alpha \quad \forall \; i = 1, 2, \ldots$

Since f continuous, $\quad f(x^{(i)}) \to f(x)$ as $x^{(i)} \to x$.

$\Rightarrow f(x) \ge \alpha \quad \Rightarrow \quad x \in D. \quad \therefore D$ is closed