Math 408 Homework 3

2.2.2 Let x_i = # lbs of product i produced by the firm

Amt of critical raw material used in the production:

$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$

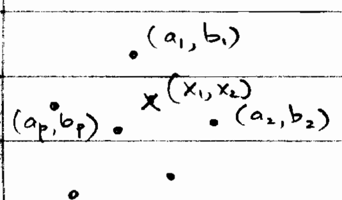
Problem

$$\text{Max} \left(S_1x_1 + S_2x_2 + S_3x_3 + S_4x_4 - \sum_{i=1}^{4} k_i x_i^2 \right)$$

$$\text{s.t} \quad \sum_{i=1}^{4} a_i x_i \le R$$

$$x_i \ge 0 \quad i = 1, \dots, 4$$

2.2.3



Distance between an arbitrary pt (x_1, x_2) and (a_i, b_i) is $\|(x_1 - a_i, x_2 - b_i)\|$

$$= \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2}$$

a) Minimize $\displaystyle\sum_{i=1}^{p} \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2}$

b) If the pt is to lie in the triangle w/ vertices $(0,0)$, $(1,0)$ & $(0,1)$

We need to Minimize $\displaystyle\sum_{i=1}^{p} \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2}$

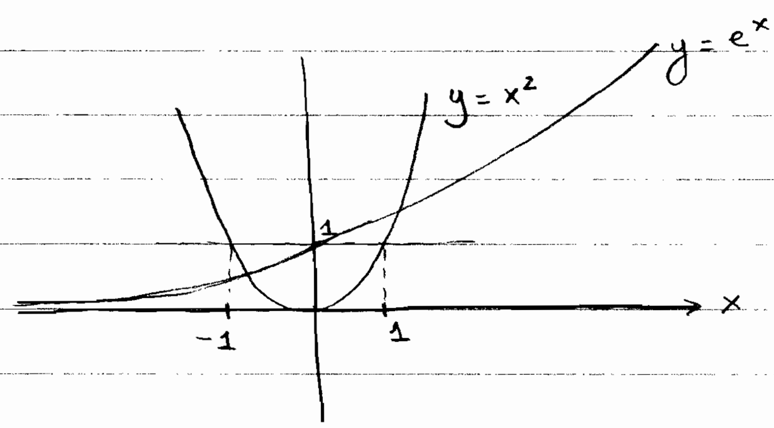
$$\text{s.t} \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_1 + x_2 \le 1$$

2.3.1 min $3x$

s.t $1 - e^x \le 0$

$x^2 - 1 \le 0$

$\Longleftrightarrow$



min $3x$

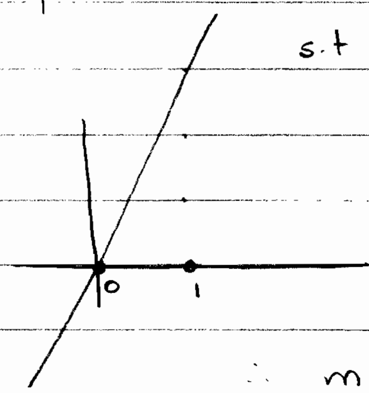
s.t $e^x \ge 1$

$x^2 \le 1$

$e^x \ge 1 \Longleftrightarrow x \ge 0 \qquad x^2 \le 1 \Longleftrightarrow -1 \le x \le 1$

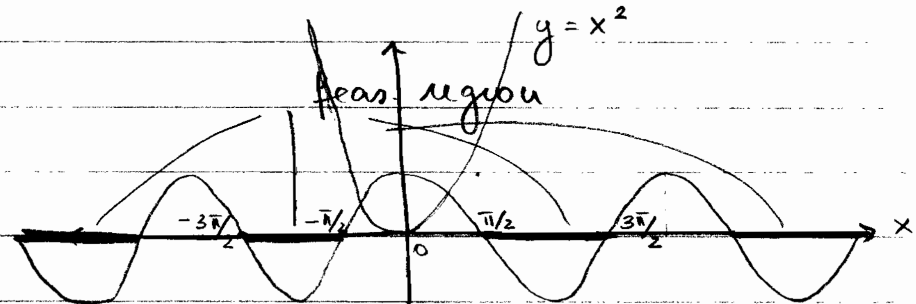
$\therefore$ problem is $\min 3x$
s.t $0 \le x \le 1$

$\boxed{\text{Global opt } (0,0) \text{ with opt value } 0}$



This is an LP – so no local opt.
It's bounded and feasible so it has a global opt.
If $0 \le x \le 1$ then $0 \le 3x \le 3$
$\therefore$ min value is 0 achieved at $x = 0$

2.3.2 $\min x^2$
$\cos(x) \le 0$



feasible region $= \bigcup_{n=0}^{\infty} \left[2n\pi + \pi/2,\ 2n\pi + \frac{3\pi}{2}\right] \cup \bigcup_{n=0}^{\infty} \left[-2n\pi - \pi/2,\ -2n\pi - 3\pi/2\right]$

From picture it looks like there are 2 local optima at $x = \pm\pi/2$ and these are also global

Check $x = \pi/2$ is local opt. Let $x \in B(\pi/2, r) \cap$ feas. region
Then $x = \pi/2 + \lambda r$ for some $0 \le \lambda < 1$, $r > 0$
$x^2 = \left(\frac{\pi}{2}\right)^2 + \lambda^2 r^2 \ge (\pi/2)^2$ $\therefore \pi/2$ is a local opt.
Similarly, $-\pi/2$ is a local opt. & they have the same cost value $= (\pi/2)^2 = \pi^2/4$

To show they are global optima, take any x in the feasible region. Then $x = \pi/2 + \lambda$ or $-\pi/2 - \lambda$ for some $\lambda \ge 0$

$$x^2 = (\pi/2 + \lambda)^2 = \frac{\pi^2}{4} + \lambda^2 + \pi\lambda \ge \frac{\pi^2}{4}$$

or $$x^2 = (-\pi/2 - \lambda)^2 = \frac{\pi^2}{4} + \lambda^2 + \pi\lambda \ge \pi^2/4$$

$\therefore$ All other sol^{n}s have higher cost.

2.3.3 min $x^5 - 5x$ feas region $= \{x \in \mathbb{R} : x \le 2\}$

s.t $x - 2 \le 0$

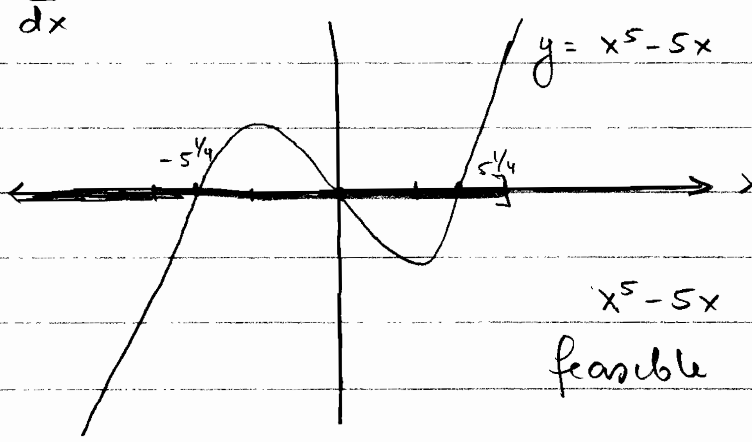
$x^5 - 5x = x(x^4 - 5) = x(x^2 - \sqrt{5})(x^2 + \sqrt{5}) = 0 \Rightarrow$

$x = 0, \; x = \pm 5^{1/4}, \; x = \pm 5^{1/4} i$ (2 complex roots, 3 real roots)

As $x \to \infty \quad x^5 - 5x \to \infty$

$x \to -\infty \quad x^5 - 5x \to -\infty$

$\frac{d}{dx} x^5 - 5x = 5x^4 - 5 = 0 \Rightarrow x^4 = 1 \quad x = \pm 1$ (critical pts)



$5^{1/4} \approx 1.495$

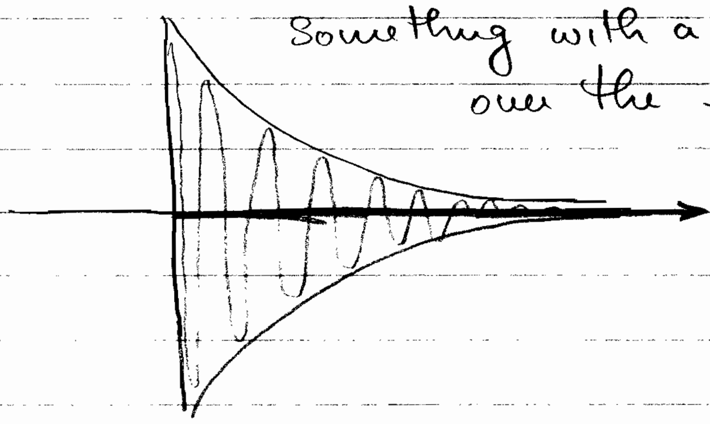
$x^5 - 5x$ is unbounded over the feasible region So no global opt

Looks like there is a local opt at $x = 1$

Check $\frac{d^2}{dx^2} x^5 - 5x = 20x^3 \qquad 20x^3 \Big|_{x=1} > 0 \Rightarrow x = 1$ local opt

2.3.4

Something with a cost function like this over the feas. region $[0, \infty)$ will work



2.3.5 $\min \; x^2 e^{-x^2} = \dfrac{x^2}{e^{x^2}} = f(x)$

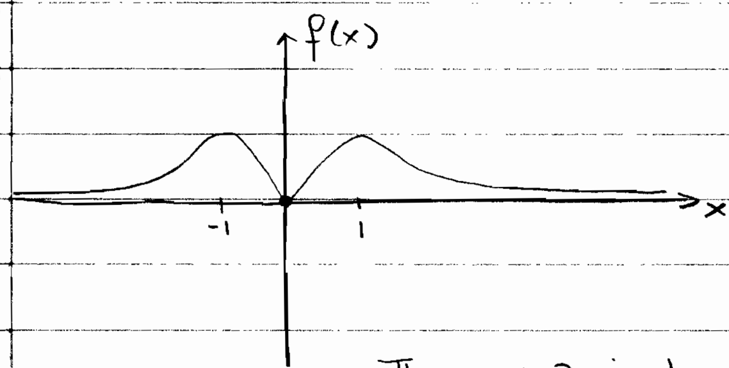
symmetric about y-axis

$f(0) = 0$

$f(x) \geq 0 \quad \forall x$

$\dfrac{d}{dx} x^2 e^{-x^2} = x^2 e^{-x^2}(-2x) + e^{-x^2}(2x) = 0$

$\Rightarrow x = 0, \pm 1$



This NLP is bounded over $\mathbb{R}$ by 0 and has a global opt at $x = 0$. It has no other local opt since at $x = \pm 1$ $\dfrac{d^2}{dx^2} f(x) < 0$

$$\frac{d^2 f(x)}{dx^2} = -4x^2 e^{-x^2} + 2(1-x^2)\left(-2x^2 e^{-x^2} + e^{-x^2}\right)$$

At $x = \pm 1$ $\dfrac{d^2 f(\pm 1)}{dx^2} = -4e^{-1} \leq 0$

2.4.1

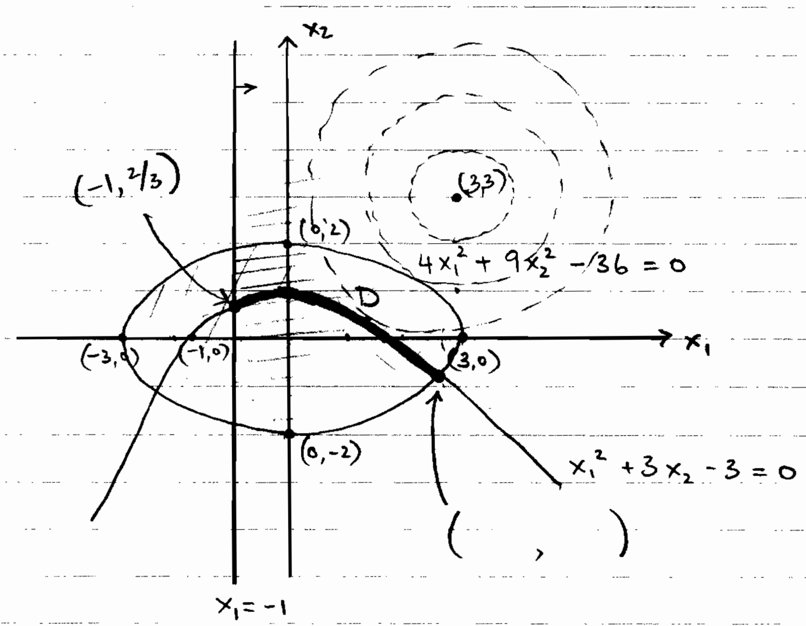
$$\min_{s.t} \; (x_1-3)^2 + (x_2-3)^2$$

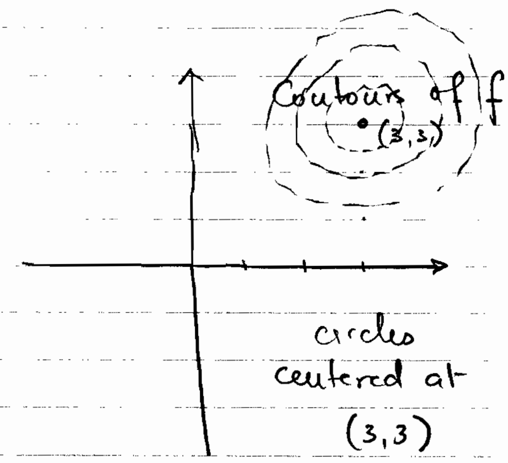
$$4x_1^2 + 9x_2^2 - 36 \le 0$$

$$-x_1 - 1 \le 0 \iff x_1 \ge -1$$

$$x_1^2 + 3x_2 - 3 = 0$$

a)





b) From the picture it looks like $\exists$ one global min somewhere in the "interior" of D. which lies on the sol^{n} set of the equation $x_1^2 + 3x_2 - 3 = 0$.
So substitute this into the objective fn to get:

$$x_2 = (3 - x_1^2)/3$$

$$f(x_1, x_2) = (x_1-3)^2 + (x_2-3)^2 = x_1^2 + 9 - 6x_1 + x_2^2 + 9 - 6x_2$$

$$= x_1^2 + 9 - 6x_1 + \left(\frac{3 - x_1^2}{3}\right)^2 + 9 - \cancel{6}^{2}\left(\frac{3 - x_1^2}{\cancel{3}}\right)$$

$$= x_1^2 + 9 - 6x_1 + \frac{9 + x_1^4 - 6x_1^2}{9} + 9 - 6 + 2x_1^2$$

$$= (9x_1^2 + 81 - 54x_1 + 9 + x_1^4 - 6x_1^2 + 81 - 54 + 18x_1^2)/9$$

$$= (x_1^4 + 21x_1^2 - 54x_1 + 117)/9$$

$$\frac{d}{dx_1}(x_1^4 + 21x_1^2 - 54x_1 + 117)/9 = \frac{4x_1^3 + 42x_1 - 54}{9} = 0$$

$$\Rightarrow x_1 = 1.143362 \qquad x_2 = \frac{3 - (1.143362)^2}{3} = 0.5642$$

$\therefore$ Candidate for global min is $(1.143362, 0.5642)$ which seems reasonable from the picture.

Check it's global min:
— First check it's a min by calculating the 2nd derivative
$\frac{d}{dx}(4x_1^3 + 42x_1 - 54) = 12x_1^2 + 42 > 0 \quad \forall x_1$

$\therefore$ The above pt is a min pt. Why is it global? There are no other critical pts, so just check the end pts and conclude that the pt we found is the lowest in cost value.

c) If we are to maximize the function, then looks like one of the end pts will be the global opt.

Calculate both end pts: $(-1, 2/3)$ and $(2.50828, -1.09716)$
f-values are: $(-4)^2 + (-7/3)^2 = 16 + \frac{49}{9} = 16 + 5.444 = 21.444$

and 17.0285
$\therefore$ The global opt appears to be $(-1, 2/3)$ //

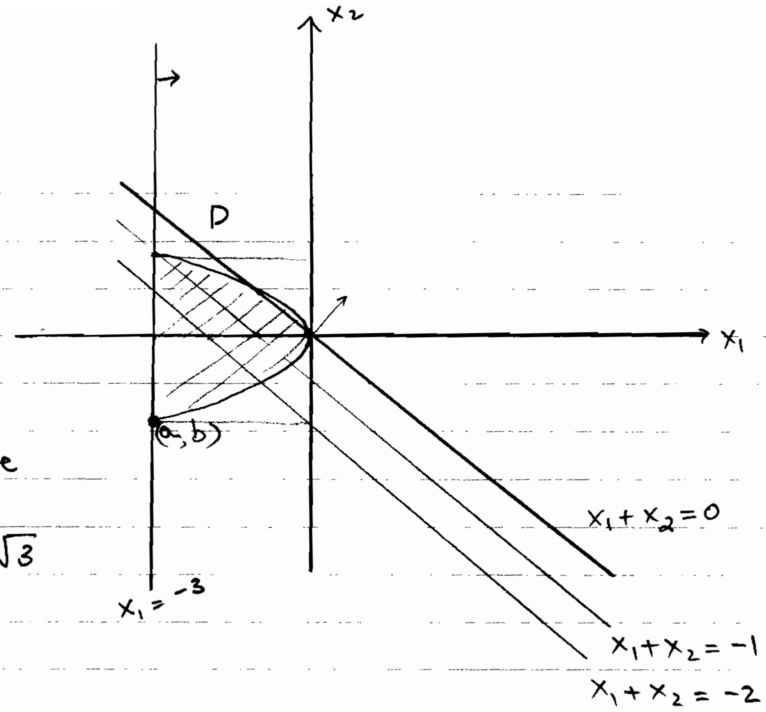
To check that this is the global min we compute critical pts of $f(x_1, x_2)$ with the substitution
$$x_2 = (3 - x_1^2)/3$$

From previous calculation there is only one critical pt and that's a min. So there are no pts in D with higher f-value than $(-1, 2/3)$ and it is the unique global max of this problem.

$\underline{2.4.2}$: $\min \quad x_1 + x_2$

s.t. $\quad x_1 + x_2^2 \le 0$

$\quad -x_1 - 3 \le 0$



a), b) Looks like the min will be at the pt (a,b)

To calculate this pt, substitute $x_1 = -3$ in $x_1 + x_2^2 = 0$ to get $-3 + x_2^2 = 0 \quad x_2 = \pm\sqrt{3}$

$\therefore (a,b) = (-3, -\sqrt{3})$

cost value $= -3-\sqrt{3}$

We need to argue that this is a global min. One way to do that is to notice that the feasible region lies in the box

$-3 \le x_1 \le 0$

$-\sqrt{3} \le x_2 \le \sqrt{3}$

Over this box, the objective $x_1 + x_2$ is minimized at $(-3, -\sqrt{3})$ which also lies in D.

$\therefore (-3, -\sqrt{3})$ is indeed the global opt.

There are no other optima

c) If we maximize $x_1 + x_2$ over D, then looks like $\exists$ a global opt on the curve $x_1 + x_2^2 = 0$.

Substituting $x_1 = -x_2^2$ into $x_1 + x_2$ get $-x_2^2 + x_2$

$\frac{d}{dx_2}(-x_2^2 + x_2) = -2x_2 + 1 = 0 \Rightarrow x_2 = \frac{1}{2}, \quad x_1 = -\frac{1}{4}$

$\therefore$ Candidate opt $(-1/4, 1/2)$

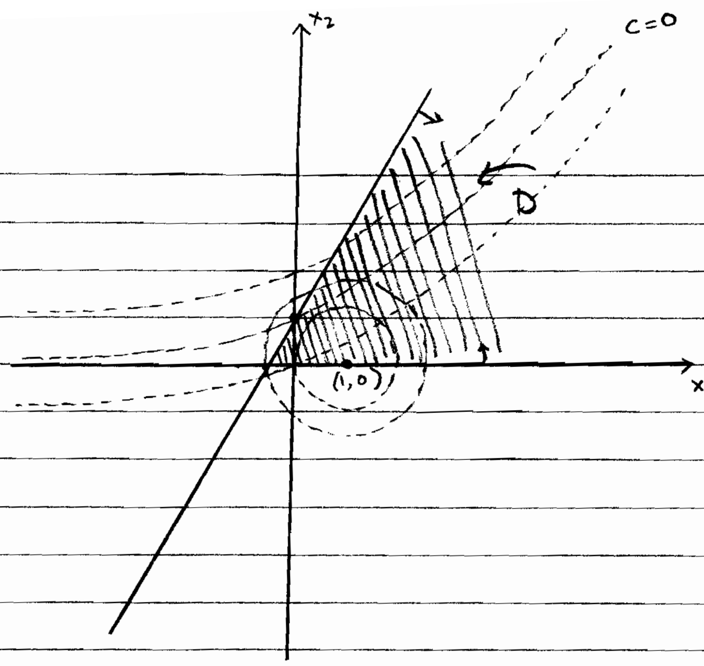
$\frac{d}{dx_2}(-2x_2 + 1) = -2 < 0 \Rightarrow$ this pt is a max. No other critical pts. So global opt $(-1/4, 1/2)$ w/ objective value $1/2$.

//

2.4.3 $\quad \min \; e^{x_1} - x_2$

$\quad$ s.t. $\; -2x_1 + x_2 \le 1$

$\quad\quad\quad x_2 \ge 0$



Contours: $\quad e^{x_1} - x_2 = 0 \Rightarrow e^{x_1} = x_2$

$\quad e^{x_1} - x_2 = 1 \Rightarrow x_2 = e^{x_1} - 1$

$\quad e^{x_1} - x_2 = -1 \Rightarrow x_2 = e^{x_1} + 1$

NLP is unbounded.

No optima

Change cost function to

$$f(x_1, x_2) = (x_1 - 1)^2 + x_2^2$$

The min value this function can take is 0 and that is attained at the point (1,0), which happen to be feasible

So the unique global min is (1,0) w/ cost value 0.

There are no local minima otherwise