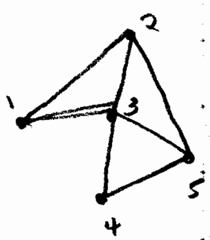
Polynomial Optimization

Ex 2 Max-cut problem in a graph



$G = (V, E)$ undirected graph with
vertex set $\quad V = \{1, \dots, n\}$
edge set $\quad E \subseteq \{ij : i, j \in V\}$
Let $w_{ij} \in \mathbb{R}$ be the weight assigned to edge $ij \in E$

Def^n: A cut in G is a partition of V into 2 sets S and $V \setminus S$.
Let $\text{cut}(S) := \{ij \in E : i \in S, j \in V \setminus S\}$ (cut edges of S)

$$\text{Weight of cut}(S) = \sum_{ij \in \text{cut}(S)} w_{ij}$$

Max-cut problem: Find the cut in G of max weight (called max-cut for short)

Want to model this problem as a NLP.

① How to model a cut mathematically?
Idea: A cut S in G can be recorded by a vector in $\mathbb{R}^n$ ($|V| = n$) with entries ± 1 where $+1$ is assigned to every vertex in S and -1 to every vertex in $V \setminus S$.
Note: $\exists$ a 1-1 correspondence between the cuts in G and vectors in $\mathbb{R}^n$ with entries ± 1.

Introduce variables x_i $i=1,..,n$ s.t x_i is assigned to node $i \in V$ and $x_i = \pm 1$
The 2^n possible values of this x-vector give the 2^n possible choices for the sets S & $V \setminus S$.

$x_i = \pm 1$ is not a valid constraint for an NLP so replace it with the equivalent condition: $x_i^2 = 1$ $i=1,..,n$

for a cut ($\therefore$ x vector with entries ± 1) the weight of the cut is $\sum_{i=1}^{n} \frac{w_{ij}}{2}(1 - x_i x_j)$
since

$$1 - x_i x_j = \begin{cases} 0 & \text{if } ij \notin cut(s) \\ 2 & \text{if } ij \in cut(s) \end{cases}$$

$\therefore$ The max-cut problem can be modeled as

$$\max \sum_{ij \in E} \frac{w_{ij}}{2}(1 - x_i x_j)$$

$$\text{s.t.} \quad x_i^2 = 1 \qquad i = 1, \dots, n$$

This is an NP-hard problem. It is also an example of a polynomial optimization problem since all functions you see in the model are polynomials.
It is in fact a quadratic (= degree 2) polynomial optimization problem.

Applications: network design, security systems etc.

Ex 3: The partition problem: Let $a_1, \ldots, a_n \in \mathbb{Z}_{>0}$. Can one partition $a_1, \ldots, a_n$ into 2 parts such that the sum of the two parts are equal?

(*) Mathematically: $\exists ?\ x_i = \pm 1 \quad i = 1, \ldots, n$ s.t $\sum_{i=1}^{n} a_i x_i = 0$

This is an example of an integer program which is a linear program in which the variables are required to be integer.

Consider the NLP:

(unconstrained polynomial opt prob)

$$\min \quad p(\underline{x}) := \left(\sum_{i=1}^{n} a_i x_i \right)^2 + \sum_{i=1}^{n} (x_i^2 - 1)^2$$

$$\text{s.t} \quad x \in \mathbb{R}^n$$

Let $p^{min} = \inf_{x \in \mathbb{R}^n} p(x)$

Claim (*) has a sol$^{\underline{n}}$ $\Longleftrightarrow$ $p^{min} = 0$.

Proof: Note first that since $x \in \mathbb{R}^n$, $p(x) \geq 0$ $\forall x \in \mathbb{R}^n$. In particular, $p^{min} \geq 0$.

(Fact to remember: Over the real numbers,
① a sum of squares is always non-negative
② If a sum of squares equals 0, then each summand is zero.)

"$\Rightarrow$" Suppose (*) has a solⁿ. — say α. Then

$$p(\alpha) = \left(\sum_{i=1}^{n} a_i \alpha_i\right)^2 + \sum_{i=1}^{n}(\alpha_i^2 - 1)^2 = 0 \quad \text{since}$$

$$\sum_{i=1}^{n} a_i \alpha_i = 0 \quad \text{and} \quad \alpha_i^2 - 1 = 0 \quad \forall\, i = 1, \ldots, n.$$

$\therefore\ 0 = p(\alpha) \leq p^{min} \Rightarrow p(\alpha) = p^{min}$ since p^{min} is the least possible value that $p(x)$ can take

$\therefore\ 0 = p^{min}$.

"$\Leftarrow$" Suppose $p^{mm} = 0$. Then $\exists\ x$ s.t

$$0 = \left(\sum_{i=1}^{n} a_i x_i\right)^2 + \sum_{i=1}^{n}(x_i^2 - 1)^2$$

(the right side is a sum of squares)

$\Rightarrow$ (by FACT mentioned earlier) each summand is 0.

$$\therefore \left(\sum_{i=1}^{n} a_i x_i\right)^2 = 0 \iff \sum_{i=1}^{n} a_i x_i = 0 \quad \text{——} (i)$$

and $x_i^2 - 1 = 0\ \forall\, i = 1, \ldots, n \iff x_i^2 = 1\ \forall\, i = 1, \ldots, n$ —— (ii)

(i) & (ii) together are exactly the constraints in (*)

$\therefore$ (*) has a solⁿ.