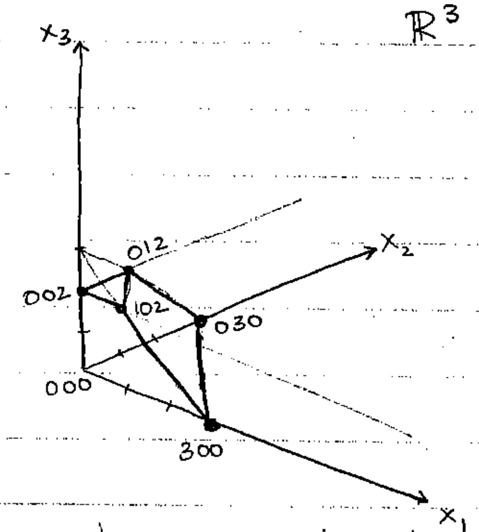
The Geometry of the Simplex Method



$$\begin{aligned}
\max \quad & x_1 + 10x_2 + 100x_3 \\
\text{s.t} \quad & x_1 + x_2 + x_3 \le 3 \\
& x_3 \le 2 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

first dictionary

$$\begin{aligned}
x_4 &= 3 - x_1 - x_2 - x_3 \\
x_5 &= 2 - x_3 \\
\hline
z &= x_1 + 10x_2 + 100x_3
\end{aligned}$$

Prob: $\max z \mid x_1, \ldots, x_5 \ge 0$

$x_c = (0,0,0,3,2)$ $\quad z = 0$

hyperplanes corresponding to the constraints:

$H_4 = H_{(x_1+x_2+x_3=3)}$, $H_{(x_3=2)} = H_5$

$H_1 = H_{(x_1=0)}$, $H_2 = H_{(x_2=0)}$, $H_{(x_3=0)} = H_3$

If $x = (x_1, \ldots, x_5)$ is a feasible sol$^{\underline{n}}$,

x_1 - distance of x from $H_{(x_1=0)}$
x_2 - " $H_{(x_2=0)}$
x_3 - " $H_{(x_3=0)}$
x_4 - " $H_{(x_1+x_2+x_3=3)}$
x_5 - " $H_{(x_3=2)}$

- basis / nonbasis
- Choose x_1 to enter the basis
  $\Rightarrow$ * allowing distance of sol$^{\underline{n}}$ from $H_{(x_1=0)}$ to increase from 0.
  * Keep $x_2 = x_3 = 0$
- Min ratio test says that x_4 leaves
  $\Rightarrow$ sol$^{\underline{n}}$ has hit the hyperplane $H_{(x_1+x_2+x_3=3)}$

  $\rightarrow$ sol$^{\underline{n}}$ travels along an edge to new basic feas. sol$^{\underline{n}}$
- new basis $\{x_1, x_5\}$ $\quad$ nonbasis $\{x_2, x_3, x_4\}$
- monotone increase.

2nd dictionary:

$$x_1 = 3 - x_2 - x_3 - x_4$$
$$x_5 = 2 - x_3$$
$$z = 3 + 9x_2 + 99x_3 - x_4$$

Choose x_3 to enter – allowing sol$^{\underline{n}}$ to leave the $H_{(x_3=0)}$

keep $x_2 = x_4 = 0$ $\therefore$ edge is uniquely chosen.

min ratio test – x_5 leaves.

– sol$^{\underline{n}}$ has hit the hyperplane $H_{(x_3=2)}$

new basis $\{x_1, x_3\}$ nonbasis $\{x_2, x_4, x_5\}$

cost has improved.

3rd dictionary

$$x_3 = 2 - x_5$$
$$x_1 = 1 - x_2 - x_4 + x_5$$
$$z = 201 + 9x_2 - 99x_5 - x_4$$

Choose x_2 to enter ... x_1 leaves.

$$x_2 = 1 - x_1 - x_4 + x_5$$
$$x_3 = 2 - x_5$$
$$z = 210 - 9x_1 - 10x_4 - 90x_5$$ ← optimal dictionary

Opt sol$^{\underline{n}}$: $(0, 1, 2, 0, 0)$ ← sol$^{\underline{n}}$ lies on the hyperplanes H_1, H_4, H_5

Q: Why are there always 3 elts in the nonbasis?

→ $x_1 \dots x_n$ measure distance from $H_1, \dots, H_n$

→ $x_{n+1} \dots x_{n+m}$ measure distance to the complicated hyperplane

<u>In general:</u>

$$\max \ c \cdot x \qquad\qquad \max \sum_{j=1}^{n} c_j x_j$$

$$Ax \le b \qquad\qquad \text{s.t} \ \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1 \dots m$$

$$x \ge 0 \qquad\qquad x_j \ge 0 \quad j = 1 \dots n$$

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$P = \left\{ x \in \mathbb{R}^n : \begin{array}{c} Ax \le b \\ x \ge 0 \end{array} \right\}$ - polyhedron / polytope if it is bounded

→ <u>convex</u>: if $a, b \in P$ then $[a,b] \subseteq P$

→ has flat "faces" or sides

→ dim (usually) n ⇒ sides have dim $n-1$

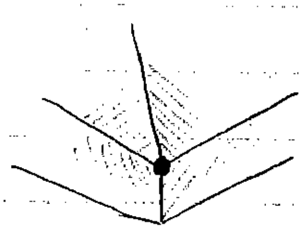
→ has faces of dim $n-1 \longrightarrow 0$ ← vertices / corners

→ Cost hyperplane gets optimized on a face → include a corner

→ Simplex method creates a monotone path (cost never becomes worse) from an initial sol^n to the opt along edges of the feasible region.

Every basic feasible sol^n has at least n vars $= 0$ since a corner lies at the intersection of at least n hyperplanes.

→ More vars can be zero:



4 hyperplanes create this corner

Some basic var is also zero. but nonbasic