Chapter 4.) The breakthrough came in 1979 when L. G. Khachian published a description of such an algorithm (based on earlier works by Shor, and by Judin and Nemirovskii). Newspapers around the world published reports of this result, some of them full of hilarious misinterpretations. We shall present the algorithm in the appendix.

For a thorough survey of the history of linear programming, the reader is referred to Chapter 2 of Dantzig's monograph (1963). References to many applications of linear programming may be found in Riley and Gass (1958). Some of the more recent applications are referenced in Gass (1975). □

PROBLEMS

Answers to problems marked with the symbol △ are found at the back of the book.

1.1 Which of the problems below are in the standard form?

a. Maximize $3x_1 - 5x_2$

subject to

$$\begin{aligned} 4x_1 + 5x_2 &\geq 3 \\ 6x_1 - 6x_2 &= 7 \\ x_1 + 8x_2 &\leq 20 \\ x_1, x_2 &\geq 0. \end{aligned}$$

b. Minimize $3x_1 + x_2 + 4x_3 + x_4 + 5x_5$

subject to

$$\begin{aligned} 9x_1 + 2x_2 + 6x_3 + 5x_4 + 3x_5 &\leq 5 \\ 8x_1 + 9x_2 + 7x_3 + 9x_4 + 3x_5 &\leq 2 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

c. Maximize $8x_1 - 4x_2$

subject to

$$\begin{aligned} 3x_1 + x_2 &\leq 7 \\ 9x_1 + 5x_2 &\leq -2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

1.2 State in the standard form:

minimize $-8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$

subject to

$$\begin{aligned} 6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 &\geq 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

1.3 Prove that (1.8) is infeasible and (1.9) is unbounded.

△**1.4** Find necessary and sufficient conditions for the numbers s and t to make the LP problem

maximize $x_1 + x_2$

subject to

$$\begin{aligned} sx_1 + tx_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a. have an optimal solution,

b. be infeasible,

c. be unbounded.

△ **1.5** Prove or disprove: If problem (1.7) is unbounded, then there is a subscript k such that the problem

$$\begin{aligned} \text{maximize} \quad & x_k \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i = 1, 2, \ldots, m) \\ & x_j \ge 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

is unbounded.

△ **1.6** [Adapted from Greene et al. (1959).] A meat packing plant produces 480 hams, 400 pork bellies, and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies, and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The *net* profits are as follows:

	Fresh	Smoked on regular time	Smoked on overtime
Hams	\$8	\$14	\$11
Bellies	\$4	\$12	\$7
Picnics	\$4	\$13	\$9

For example, the following schedule yields a total net profit of \$9,965:

	Fresh	Smoked	Smoked (overtime)
Hams	165	280	35
Bellies	295	70	35
Picnics	55	70	105

The objective is to find the schedule that maximizes the total net profit. Formulate as an LP problem in the standard form.

1.7 [Adapted from Charnes et al. (1952).] An oil refinery produces four types of raw gasoline: alkylate, catalytic-cracked, straight-run, and isopentane. Two important characteristics of each gasoline are its performance number PN (indicating antiknock properties) and its vapor pressure RVP (indicating volatility). These two characteristics, together with the production levels in barrels per day, are as follows:

	PN	RVP	Barrels produced
Alkylate	107	5	3,814
Catalytic-cracked	93	8	2,666
Straight-run	87	4	4,016
Isopentane	108	21	1,300

These gasolines can be sold either raw, at \$4.83 per barrel, or blended into aviation gasolines (Avgas A and/or Avgas B). Quality standards impose certain requirements on the aviation gasolines; these requirements, together with the selling prices, are as follows:

	PN	RVP	Price per barrel
Avgas A	at least 100	at most 7	\$6.45
Avgas B	at least 91	at most 7	\$5.91

The PN and RVP of each mixture are simply weighted averages of the PNs and RVPs of its constituents. For example, the refinery could adopt the following strategy:

- Blend 2,666 barrels of alkylate and 2,666 barrels of catalytic into 5,332 barrels of Avgas A with

$$\mathrm{PN} = \frac{(2{,}666 \times 107) + (2{,}666 \times 93)}{5{,}332} = 100$$

$$\mathrm{RVP} = \frac{(2{,}666 \times 5) + (2{,}666 \times 8)}{5{,}332} = 6.5.$$

- Blend 1,148 barrels of alkylate, 4,016 barrels of straight-run, and 1,024 barrels of isopentane into 6,188 barrels of Avgas B with

$$\mathrm{PN} = \frac{(1{,}148 \times 107) + (4{,}016 \times 87) + (1{,}024 \times 108)}{6{,}188} \doteq 94.2$$

$$\mathrm{RVP} = \frac{(1{,}148 \times 5) + (4{,}016 \times 4) + (1{,}024 \times 21)}{6{,}188} \doteq 7.$$

  Sell 276 barrels of isopentane raw.

This sample plan yields a total profit of

$$(5{,}332 \times 6.45) + (6{,}188 \times 5.91) + (276 \times 4.83) \doteq \$72{,}296.$$

The refinery aims for the plan that yields the largest possible profit. Formulate as an LP problem in the standard form.

1.8 An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay \$20 for each radio delivered by the end of the first week, \$18 for those delivered by the end of the second week, \$16 by the end of the third week, and \$14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll till the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are \$200; the weekly wages of a trainee are \$100. The production costs, excluding the worker's wages, are \$5 per radio.

For example, the company could adopt the following program.

First week: 10 assemblers, 30 instructors, 90 trainees
Workers' wages: \$8,000

	Trainees' wages: \$9,000 Profit from 500 radios: \$7,500 Net loss: \$9,500
Second week:	120 assemblers, 10 instructors, 30 trainees Workers' wages: \$26,000 Trainees' wages: \$3,000 Profit from 6,000 radios: \$78,000 Net profit: \$49,000
Third week:	160 assemblers Workers' wages: \$32,000 Profit from 8,000 radios: \$88,000 Net profit: \$56,000
Fourth week:	110 assemblers, 50 idle Workers' wages: \$32,000 Profit from 5,500 radios: \$49,500 Net profit: \$17,500

This program, leading to a total net profit of \$113,000, is one of many possible programs. The company's aim is to maximize the total net profit. Formulate as an LP problem (not necessarily in the standard form).

△ **1.9** [S. Masuda (1970); see also V. Chvátal (1983).] The *bicycle problem* involves n people who have to travel a distance of ten miles, and have one single-seat bicycle at their disposal. The data are specified by the walking speed w_j and the bicycling speed b_j of each person j ($j = 1, 2, \ldots, n$); the task is to minimize the arrival time of the last person. (Can you solve the case of $n = 3$ and $w_1 = 4$, $w_2 = w_3 = 2$, $b_1 = 16$, $b_2 = b_3 = 12$?) Show that the optimal value of the LP problem

$$\begin{array}{lrl}
\text{minimize} & t & \\
\text{subject to} & t - x_j - x'_j - y_j - y'_j \geq 0 & (j = 1, 2, \ldots, n) \\
& t - \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} y'_j \geq 0 & \\
& w_j x_j - w_j x'_j + b_j y_j - b_j y'_j = 10 & (j = 1, 2, \ldots, n) \\
& \sum_{j=1}^{n} b_j y_j - \sum_{j=1}^{n} b_j y'_j \leq 10 & \\
& x_j, x'_j, y_j, y'_j \geq 0 & (j = 1, 2, \ldots, n)
\end{array}$$

provides a lower bound on the optimal value of the bicycle problem.