

Math 308 N Test Prep Homework
DUE FRIDAY MAR 15

- (1) (Practice showing that something is a subspace). Suppose λ is an eigenvalue for the matrix A . Let S be the λ -eigenspace:

$$S = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\},$$

the set of all vectors \vec{v} satisfying the equation $A\vec{v} = \lambda\vec{v}$. In class, we said (roughly) that S is a subspace because $S = \text{null}(A - \lambda I)$. For this problem, instead show that S is a subspace by checking the three conditions on S .

(2) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

- (a) Compare the eigenvalues and eigenvectors of A , A^2 and A^{-1} . Are they similar or different?
- (b) Same questions as (a) but for A, B and AB .
- (c) Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. Then, compute A^{1000} . (Hint: you can use the method from the bonus problem of Midterm #2.)
- (c) Suppose \vec{v} is an eigenvector of an arbitrary matrix M , with eigenvalue λ . Show (using matrix algebra) that \vec{v} is also an eigenvector of $M + I$, but with a different eigenvalue. What eigenvalue is it?

- (3) (Reflections and projections)

- (a) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the transformation from Worksheet 4, problem 1:

$$T(\vec{x}) = \frac{1}{3} \begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x}.$$

Determine the eigenvalues of T , and find a basis for each eigenspace.

Note: You should find that the eigenvalues of T are 1 and -1 .

- (b) Remember that T is supposed to be ‘reflection across a plane S ’. Explain what the eigenvalues and eigenvectors from (a) mean geometrically. What is their relationship to S ? Why does it make sense for the eigenvalues to be 1, -1 ?
- (c) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the “averaging transformation”:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \end{bmatrix}.$$

Find all eigenvalues and eigenspaces for T . Explain your answer (what does it mean in terms of ‘averaging’?)

(4) (Rotations)

(a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by $\pi/3$. Compute the characteristic polynomial of T , and find any eigenvalues and eigenvectors. (You can look up the matrix for T from previous worksheets or your notes from class).

(b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a rotation in \mathbb{R}^3 by $\pi/3$ around some chosen axis L , a line through the origin in \mathbb{R}^3 . **Without computing any matrices**, explain why $\lambda = 1$ is always an eigenvalue of T . What is the corresponding eigenspace?

(5) Find a 3×3 matrix A with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with $\lambda = 1$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ with

$\lambda = 2$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda = 10$. You may take for granted that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a

basis for \mathbb{R}^3 .

(**Hint:** A must be diagonalizable, $A = PDP^{-1}$. Figure out P and D , then compute A directly.)