Practice Quiz 2 – Math 308 E

Consider the following matrix A and its echelon form B:

$$A = \begin{pmatrix} 16 & 22 & 18 & 18 \\ 2 & 4 & 0 & 2 \\ 8 & 4 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & 0 & 2 \\ 0 & -10 & 18 & 2 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

Suppose T is the linear transformation that sends $\mathbf{x}$ to $A\mathbf{x}$.

(1) (a) Solve $Ax = 0$.
 (b) Express the set of solutions to $Ax = 0$ as the span of a set of vectors.
 (c) Recall that a collection of vector $u_1, \ldots, u_m \in \mathbb{R}^n$ is linearly independent if and only if the only solution to $u_1x_1 + \cdots + u_mx_m = 0$ is $x_1 = 0, \ldots, x_m = 0$. Using this check, show that the vectors you computed in the previous step are linearly independent.
 (d) Compute a basis for the nullspace of A.
 (e) Compute the nullity of A.
 (f) Do A and B have the same nullspaces?
 (g) What does the number of free variables in the solution space of $Ax = 0$ have to do with the nullspace of A?

(2) (a) What is the domain and codomain of T?
 (b) Is T one-to-one? Explain.

(3) (a) Where does the column space of A lie?
 (b) Find a basis for the column space of A.
 (c) Is the first column of B in the column space of A?
 (d) Is the sum of the first and third columns of A in the column space of A?
 (e) Are the column spaces of A and B the same?
 (f) What is the dimension of the column space of A?
 (g) Check that the pivot columns of B are linearly independent.
 (h) Are the pivot columns of any matrix in echelon form linearly independent? Why?
 (i) Why are the columns of A corresponding to the pivot columns of B linearly independent?

(4) (a) What is the range of T?
 (b) Is the range of T equal to the row space or column space or nullspace of A?
 (c) What is the dimension of the range of T?
 (d) Is the 0 vector always in the range of T? Why?
 (e) Is the 0 vector always in the nullspace of T? Why?
 (f) If $Ax = 0$ has infinitely many solutions, then can T be one-to-one? Explain/give a counterexample.
 (g) If $Ax = 0$ has infinitely many solutions, then can T be onto? Explain/give a counterexample.

(5) (a) If $Ax = b$ has a solution, must b lie in the range of T?
 (b) If $Ax = b$ has a solution, must b lie in the row space/column space/nullspace of A?
 (c) Does $Ax = 0$ always have a solution?

(6) (a) Do A and B have the same row spaces?
 (b) Find a basis for the row space of A.
 (c) Find a basis for the row space of B.
 (d) Are the pivot rows of B linearly independent?
 (e) Are the corresponding rows of A linearly independent?
 (f) In general, are the pivot rows of a matrix in echelon form linearly independent?

    (g) Does the rowspace of A have anything to do with the nullspace or column space of T?

(7) (a) What is the rank of A?
    (b) What is the rank of B?

(8) (a) Suppose T' is the another linear transformation that sends x to Bx.
    (b) Is range of T equal to the range of T'?
    (c) Is kernel of T equal to the kernel of T'?
    (d) Is it true that T is onto if and only if T' is onto?
    (e) Is it true that T is one-to-one if and only if T' is one-to-one?
    (f) Are T and T' the same linear transformations?

(9) (a) Suppose C is the 3×3 matrix consisting of the first three columns of A. Is C invertible?
    (b) If yes, find C^{-1}.
    (c) What is the rank of C?
    (d) How is the column space of C related to the column space of A?
    (e) How is the row space of C related to the row space of A?
    (f) How is the nullspace of C related to the nullspace of A?
    (g) Is $\{(1,0,0),(0,1,0),(0,0,1)\}$ a basis for the row space/column space/nullspace of C?
    (h) Does $Cx = b$ have a solution for any $b \in \mathbb{R}^3$? If so, find a solution.
    (i) How many solutions are there?
    (j) What is the nullity of C?