Practice Quiz – Math 308 E – SOLUTIONS

Recall that if we have m vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^n$, then we can form the matrix A whose columns are $u_1, \ldots, u_m$. Let B be the echelon form of A. All the questions below are based on such a matrix B. Most questions have a yes/no answer, but I am mostly interested in your reasons for the answer. Give full reasons for all answers.

Suppose we are given the following matrix B:

$$\begin{pmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since B is the echelon form of A, both A and B have the same size – 3 rows and 4 columns.

(1) What is n?
    $n = 3$. This is the size of each u_i.

(2) What is m?
    $m = 4$. This is the number of vectors u_i.

(3) Are $u_1, \ldots, u_m$ linearly independent?
    No, they are not since B does not have a pivot in every column. So there would be free variables when you solve $Ax = 0$ or equivalently, $Bx = 0$ allowing for non-zero solutions to $Ax = 0$.

(4) Does $\{u_1, \ldots, u_m\}$ span $\mathbb{R}^n$?
    No, since B does not have a pivot in every row. If you want to solve $Ax = b$, then the echelon form of the augmented matrix $[A \,|\, b]$ would be $[B \,|\, b']$ for some vector b'. Since the last row of B is all zeros, you would need the last entry of b' to also be zero for $Ax = b$ to have a solution. So not all b in $\mathbb{R}^3$ is in the span of u_1, u_2, u_3, u_4.

(5) Looking at B can you write down a subset of the original set $\{u_1, \ldots, u_m\}$ that would be guaranteed to be linearly independent?
    $\{u_1, u_3\}$ are linearly independent. This is because the echelon form of $[u_1, u_3]$ consists of the first and third column of B which has a pivot in every column.

(6) Is there a subset of the original set $\{u_1, \ldots, u_m\}$ that would be guaranteed to span $\mathbb{R}^n$?
    No. Recall that the span of a collection of vectors is the set of all linear combinations of the vectors. The original set $\{u_1, \ldots, u_4\}$ already did not span $\mathbb{R}^3$, so a subset cannot either.

(7) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ does not have a solution.
    Pick a b with non-zero last coordinate. Example: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then the last row of $[B \,|\, b]$ says $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$ which does not have a solution.

(8) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ has a solution.
    Pick a b with zero last coordinate. Example: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(9) Write down a $b \in \mathbb{R}^n$ for which $Bx = b$ has a unique solution.
    There is no such b since there are free variables in the solution of $Bx = b$ when there is a solution.

(10) Is there a new vector $w \in \mathbb{R}^n$ that you could add to the set $\{u_1, \ldots, u_m\}$ to guarantee that $\{u_1, \ldots, u_m, w\}$ will span $\mathbb{R}^n$?

Yes, pick $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then the echelon form of A (assuming that you did not switch any rows while doing Gaussian elimination) will still be this vector. Now there is a pivot in every row of the echelon form and the five vectors together will span $\mathbb{R}^3$.

(11) Is there a column of B that is in the span of the rest? If so, find it.

Yes. The columns of B are linearly dependent since not all columns of B have a pivot. Solving $Bx = 0$ we see that x_2 and x_4 are free variables. Set $x_4 = 2$ and $x_2 = 1$. Then we get the solution $x_1 = -3$ and $x_3 = 1$. This means that $-3B_1 + B_2 + B_3 + 2B_4 = 0$ where B_i is the ith column of B. So any column is in the span of the others.

(12) Looking at B do you see a u_i that is in the span of the others? How can you identify it?

Remember that $Ax = 0$ and $Bx = 0$ have the same solutions. So the above solution $(-3, 1, 1, 2)$ is also a solution of $Ax = 0$. By the same logic as above, every column of A is the span of the others.

(13) Assuming that no row of A was a zero row, how many planes are being used to cut out the solutions of $Ax = 0$?

Three planes, each given by a row of A – i.e., the normal vector of each plane is a row of A.

(14) Looking at B, how many planes are needed at the minimum to cut out the solution set of $Ax = 0$?

Two planes, with normal vectors given by the two non-zero rows of B.

(15) If your answers to the last two questions are different, why is there a difference?

There is a difference because Gaussian elimination tells us that the last row of B is a row of zeros. Therefore, the last row of A was in fact a linear combination of the first two rows of A.

(16) Put B into reduced echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(17) Write down a non-zero solution of $Ax = 0$ if you can.

We already saw one in (12). From the reduced echelon form you can get the solutions $x_3 = \frac{1}{2}x_4$ and $x_1 = -\frac{3}{2}x_4$. Choose $x_4 = 1$ say and you get the solution $(-3/2, 0, 1/2, 1)$.

(18) How many free variables are there in the set of solutions to $Ax = b$ when there is a solution?

Two – x_2 and x_4.

(19) If you erased the last row of zeros in B then would the columns of the resulting matrix be linearly independent?

No. Some columns are still missing a pivot.

(20) Can you add rows to B to make the columns of the new matrix linearly independent? If yes, give an example of the new matrix you would construct.

Yes. You need to add enough rows to get a pivot in every column. For example:

$$\begin{pmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$