THE STRUCTURE OF GROUP RELAXATIONS

REKHA R. THOMAS

Contents

1. Introduction

Group relaxations of integer programs were introduced by Ralph Gomory in the 1960s [15], [17]. Given a general integer program of the form

$$\text{(1)} \qquad \textit{minimize } \{c \cdot x \ : \ Ax = b, \ x \geq 0, \ \textit{integer}\},$$

its group relaxation is obtained by dropping non-negativity restrictions on all the basic variables in the optimal solution of its linear relaxation. In this article, we survey recent results on group relaxations obtained from the algebraic study of integer programming using *Gröbner bases* of *toric ideals* [39]. No knowledge of these methods is assumed, and the exposition is self-contained and hopefully accessible to a person familiar with the traditional methods of integer programming. For the reader who might be interested in the algebraic origins, motivations and counterparts of the described results, we have included brief comments in the last section. These comments are numbered in the style of footnotes and organized as paragraphs in Section 8. While they offer a more complete picture of the theory to those familiar with commutative algebra, they are not necessary for the continuity of the article.

For the sake of brevity, we will bypass a detailed account of the classical theory of group relaxations. A short expository account can be found in [36, §24.2], and a detailed set of lecture notes on this topic in [28]. We give a brief synopsis of the essentials based on the recent survey article by Aardal et. al [3] and refer the reader to any of the above sources for further details and references on the classical theory of group relaxations.

Date: November 12, 2002.
Research partially supported by NSF grant DMS-0100141.

Assuming that all data in (1) are integral and that A_B is the optimal basis of the linear relaxation of (1), Gomory's group relaxation of (1) is the problem

$$(2) \qquad minimize\ \{\tilde{c}_N \cdot x_N\ :\ A_B^{-1}A_N x_N \equiv A_B^{-1}b\ (mod\ 1),\ x_N \geq 0,\ integer\}.$$

Here B and N are the index sets for the basic and non-basic columns of A corresponding to the optimal solution of the linear relaxation of (1). The vector x_N denotes the non-basic variables and the cost vector $\tilde{c}_N = c_N - c_B A_B^{-1} A_N$ where $c = (c_B, c_N)$ is partitioned according to B and N. The notation $A_B^{-1}A_N x_N \equiv A_B^{-1}b\ (mod\ 1)$ indicates that $A_B^{-1}A_N x_N - A_B^{-1}b$ is a vector of integers. Problem (2) is called a "group relaxation" of (1) since it can be written in the canonical form

$$(3) \qquad minimize\ \{\tilde{c}_N \cdot x_N\ :\ \sum_{j\in N} g_j x_j \equiv g_0\ (mod\ G),\ x_N \geq 0,\ integer\}$$

where G is a finite abelian group and $g_j \in G$. Problem (3) can be viewed as a shortest path problem in a graph on $|G|$ nodes which immediately furnishes algorithms for solving it. Once the optimal solution x_N^* of (2) is found, it can be uniquely *lifted* to a vector $x^* = (x_B^*, x_N^*) \in \mathbb{Z}^n$ such that $Ax^* = b$. If $x_B^* \geq 0$ then x^* is the optimal solution of (1). Otherwise, $c \cdot x^*$ is a lower bound for the optimal value of (1). Several strategies are possible when the group relaxation fails to solve the integer program. See [5],[20], [34] and [46] for work in this direction. A particular idea due to Wolsey [45], that is very relevant for this paper, is to consider the *extended* group relaxations of (1). These are all the possible group relaxations of (1) obtained by dropping non-negativity restrictions on all possible subsets of the basic variables x_B in the optimum of the linear relaxation of (1). Gomory's group relaxation (2) of (1) and (1) itself are therefore among these extended group relaxations. If (2) does not solve (1), then one could resort to other extended relaxations to solve the problem. At least one of these extended group relaxations (in the worst case (1) itself) is guaranteed to solve the integer program (1).

The convex hull of the feasible solutions to (2) is called the *corner polyhedron* [16]. A major focus of Gomory and others who worked on group relaxations was to understand the polyhedral structure of the corner polyhedron. This was achieved via the *master polyhedron* of the group G [17] which is the convex hull of the set of points

$$\{z\ :\ \sum_{g\in G} g z_g \equiv g_0\ (mod\ G),\ z \geq 0,\ integer\}.$$

Facet-defining inequalities for the master polyhedron provide facet inequalities of the corner polyhedron [17]. As remarked in [3], this landmark paper [17] introduced several of the now standard ideas in polyhedral combinatorics like projection onto faces, subadditivity, master polytopes, using automorphisms to generate one facet from another, lifting techniques and so on. See [19] for further results on generating facet inequalities. Recent results on the facets of master polyhedra and cutting planes can be found in [4], [12] and [18].

In the algebraic approach to integer programming, one considers the entire family of integer programs of the form (1) as the right hand side vector b varies. Definition 2.6 defines a set of group relaxations for each program in this family. Each relaxation is indexed by a face of a simplicial complex called a *regular triangulation* (Definition 2.1). This complex encodes all the optimal bases of the linear programs arising from the coefficient matrix A

and cost vector c (Lemma 2.3). The main result of Section 2 is Theorem 2.8 which states that the group relaxations in Definition 2.6 are precisely all the bounded group relaxations of all programs in the family. In particular, they include all the extended group relaxations of all programs in the family and typically contain more relaxations for each program. This theorem is proved via a particular reformulation of group relaxations which is crucial for the rest of the paper. This and other reformulations are described in Section 2.

The most useful group relaxations of an integer program are the "least strict" ones among all those that solve the program. By this we mean that any further relaxation of non-negativity restrictions will result in group relaxations that do not solve the problem. The faces of the regular triangulation indexing all these special relaxations for all programs in the family are called the *associated sets* of the family (Definition 3.1). In Section 3 we develop tools to study associated sets. This leads to Theorem 3.11 which characterizes associated sets in terms of *standard pairs* and *standard polytopes*. Theorem 3.12 shows that one can "read off" the "least strict" group relaxations that solve a given integer program in the family from these standard pairs.

The results in Section 3 lead to an important invariant of the family of integer programs being studied called its *arithmetic degree*. In Section 4 we discuss the relevance of this invariant and give a bound for it based on a result of Ravi Kannan (Theorem 4.8). His result builds a bridge between our methods and those of Kannan, Lenstra, Lovasz, Scarf and others that use geometry of numbers in integer programming.

Section 5 examines the structure of the poset of associated sets. The main result in this section is the *chain theorem* (Theorem 5.2) which shows that associated sets occur in saturated chains. Theorem 5.4 bounds the length of a maximal chain.

In Section 6 we define a particular family of integer programs called a *Gomory family*, for which all associated sets are maximal faces of the regular triangulation. Theorem 6.2 gives several characterizations of Gomory families. We show that this notion generalizes the classical notion of *total dual integrality* in integer programming [36, §22]. We conclude in Section 7 with constructions of Gomory families from matrices whose columns form a Hilbert basis. In particular, we recast the existence of a Gomory family as a *Hilbert cover* problem. This builds a connection to the work of Sebö [37], Bruns & Gubeladze [8] and Firla & Ziegler [13] on *Hilbert partitions* and *covers* of polyhedral cones. We describe the notions of *super* and Δ-*normality* both of which give rise to Gomory families (Theorems 7.8 and 7.15).

The majority of the material in this paper is a translation of algebraic results from [24], [25], [26], [39, §8 and §12.D], [41] and [42]. The translation has sometimes required new definitions and proofs. Kannan's theorem in Section 4 has not appeared elsewhere.

We will use the letter $\mathbb{N}$ to denote the set of non-negative integers, $\mathbb{R}$ to denote the real numbers and $\mathbb{Z}$ for the integers. The symbol $P \subseteq Q$ denotes that P is a subset of Q, possibly equal to Q, while $P \subset Q$ denotes that P is a proper subset of Q.

2. Group Relaxations

Throughout this article, we fix a matrix $A \in \mathbb{Z}^{d\times n}$ of rank d, a cost vector $c \in \mathbb{Z}^n$ and consider the family $IP_{A,c}$ of all integer programs

$$IP_{A,c}(b) := minimize\ \{c \cdot x\ :\ Ax = b,\ x \in \mathbb{N}^n\}$$

as b varies in the semigroup $\mathbb{N}A := \{Au : u \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$. This family is precisely the set of all feasible integer programs with coefficient matrix A and cost vector c. The semigroup $\mathbb{N}A$ lies in the intersection of the d-dimensional polyhedral cone $cone(A) := \{Au : u \geq 0\} \subseteq \mathbb{R}^d$ and the d-dimensional lattice $\mathbb{Z}A := \{Au : u \in \mathbb{Z}^n\} \subseteq \mathbb{Z}^d$. For simplicity, we will assume that $cone(A)$ is pointed and that $\{u \in \mathbb{R}^n : Au = 0\}$, the kernel of A, intersects the non-negative orthant of $\mathbb{R}^n$ only at the origin. This guarantees that all programs in $IP_{A,c}$ are bounded. In addition, the cost vector c will be assumed to be *generic* in the sense that each program in $IP_{A,c}$ has a unique optimal solution. The algebraic study of integer programming shows that all cost vectors in $\mathbb{R}^n$ except those on (parts of) a finite number of hyperplanes, are generic for the family $IP_{A,c}$ [40]. Hence the genericity assumption on c is almost always satisfied. In fact all cost vectors can be made generic by breaking ties with a fixed total order on $\mathbb{N}^n$ such as the lexicographic order. Geometrically, this has the effect of perturbing a non-generic c to a vector that no longer lies on one of the forbidden hyperplanes, while keeping the optimal value of the programs in $IP_{A,c}$ unchanged.

The *linear relaxation* of $IP_{A,c}(b)$ is the linear program

$$LP_{A,c}(b) := minimize \ \{ \ c \cdot x \ : \ Ax = b, \ x \geq 0\}.$$

We denote by $LP_{A,c}$ the family of all linear programs of the form $LP_{A,c}(b)$ as b varies in $cone(A)$. These are all the feasible linear programs with coefficient matrix A and cost vector c. Since all data are integral and all programs in $IP_{A,c}$ are bounded, all programs in $LP_{A,c}$ are bounded as well.

In the classical definitions of group relaxations of $IP_{A,c}(b)$, one assumes knowledge of the optimal basis of the linear relaxation $LP_{A,c}(b)$. In the algebraic set up, we define group relaxations for all members of $IP_{A,c}$ at one shot and, analogously to the classical setting, assume that the optimal bases of all programs in $LP_{A,c}$ are known. This information is carried by a *polyhedral complex* called the *regular triangulation* of $cone(A)$ with respect to c.

A polyhedral complex Δ is a collection of polyhedra called *cells (or faces)* of Δ such that:
(i) every face of a cell of Δ is again a cell of Δ and,
(ii) the intersection of any two cells of Δ is a common face of both.
The set-theoretic union of the cells of Δ is called the *support* of Δ. If Δ is not empty, then the empty set is a cell of Δ since it is a face of every polyhedron. If all the faces of Δ are cones, we call Δ a *cone complex.*

For $\sigma \subseteq \{1, \ldots, n\}$, let A_σ be the submatrix of A whose columns are indexed by σ, and let $cone(A_\sigma)$ denote the cone generated by the columns of A_σ. The *regular subdivision* Δ_c of $cone(A)$ is a cone complex with support $cone(A)$ defined as follows.

Definition 2.1. For $\sigma \subseteq \{1, \ldots, n\}$, $cone(A_\sigma)$ is a face of the *regular subdivision* Δ_c of $cone(A)$ if and only if there exists a vector $y \in \mathbb{R}^d$ such that $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j < c_j$ for all $j \notin \sigma$.

The regular subdivision Δ_c can be constructed geometrically as follows. Consider the cone in $\mathbb{R}^{d+1}$ generated by the lifted vectors $(a_i^t, c_i) \in \mathbb{R}^{d+1}$ where a_i is the ith column of A and c_i is the ith component of c. The *lower* facets of this lifted cone are all those facets whose normal vectors have a negative $(d+1)$th component. Projecting these lower facets back onto $cone(A)$ induces the regular subdivision Δ_c of $cone(A)$ (see [6]). Note that if the columns

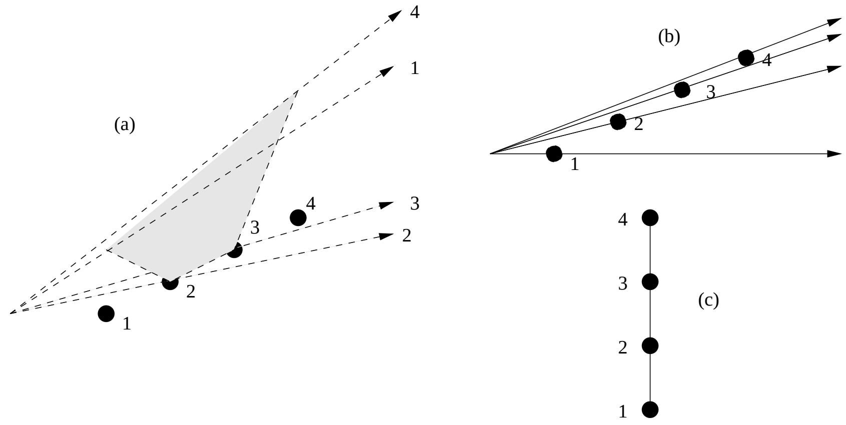

FIGURE 1. Regular triangulation Δ_c for $c = (1, 0, 0, 1)$ (Example 2.2 (i)).

of A span an affine hyperplane in $\mathbb{R}^d$, then Δ_c can also be seen as a subdivision of $conv(A)$, the $(d-1)$-dimensional convex hull of the columns of A.

The genericity assumption on c implies that Δ_c is in fact a triangulation of $cone(A)$ (see [40]). We call Δ_c the *regular triangulation* of $cone(A)$ with respect to c. For brevity, we may also refer to Δ_c as the regular triangulation of A with respect to c. Using σ to label $cone(A_\sigma)$, Δ_c is usually denoted as a set of subsets of $\{1, \ldots, n\}$. Since Δ_c is a complex of simplicial cones, it suffices to list just the maximal elements (with respect to inclusion) in this set of sets. By definition, every one dimensional face of Δ_c is of the form $cone(a_i)$ for some column a_i of A. However, not all cones of the form $cone(a_i)$, a_i a column of A, need appear as a one dimensional cell of Δ_c.

Example 2.2. (i) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $c = (1, 0, 0, 1)$. The four columns of A are the four dark points in Figure 1 labeled by their column indices $1, \ldots, 4$. Figure 1 (a) shows the cone generated by the lifted vectors $(a_i^t, c_i) \in \mathbb{R}^3$. The rays generated by the lifted vectors have the same labels as the points that were lifted. Projecting the lower facets of this lifted cone back onto $cone(A)$, we get the regular triangulation Δ_c of $cone(A)$ shown in Figure 1 (b). The same triangulation is shown as a triangulation of $conv(A)$ in Figure 1 (c). The faces of the triangulation Δ_c are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\emptyset$. Using only the maximal faces, we may write $\Delta_c = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$.
(ii) For the A in (i), $cone(A)$ has four distinct regular triangulations as c varies. For instance, the cost vector $c' = (0, 1, 0, 1)$ induces the regular triangulation $\Delta_{c'} = \{\{1, 3\}, \{3, 4\}\}$ shown in Figure 2 (b) and (c). Notice that $\{2\}$ is not a face of $\Delta_{c'}$.
(iii) If $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $c = (1, 0, 0, 1)$, then $\Delta_c = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. However, in this case, Δ_c can only be seen as a triangulation of $cone(A)$ and not of $conv(A)$. $\square$

For a vector $x \in \mathbb{R}^n$, let $supp(x) := \{i : x_i \neq 0\}$ denote the *support* of x. The significance of regular triangulations for linear programming is summarized in the following proposition.

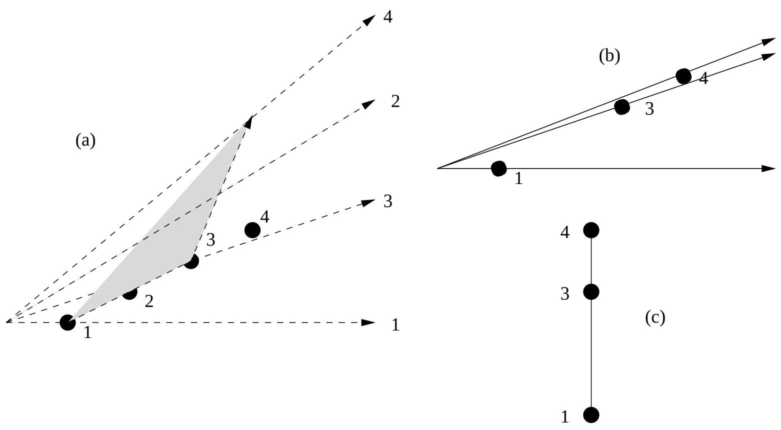

FIGURE 2. Regular triangulation $\Delta_{c'}$ for $c' = (0,1,0,1)$ (Example 2.2 (ii)).

Proposition 2.3. [40, Lemma 1.4] *An optimal solution of $LP_{A,c}(b)$ is any feasible solution x^* such that $supp(x^*) = \tau$ where τ is the smallest face of the regular triangulation Δ_c such that $b \in cone(A_\tau)$.*

Proposition 2.3 implies that $\sigma \subseteq \{1,\ldots,n\}$ is a maximal face of Δ_c if and only if A_σ is an optimal basis for all $LP_{A,c}(b)$ with b in $cone(A_\sigma)$. For instance, in Example 2.2 (i), if $b = (4,1)^t$ then the optimal basis of $LP_{A,c}(b)$ is $[a_1,a_2]$ where as if $b = (2,2)^t$, then the optimal solution of $LP_{A,c}(b)$ is degenerate and either $[a_1,a_2]$ or $[a_2,a_3]$ could be the optimal basis of the linear program. (Recall that a_i is the ith column of A.) All programs in $LP_{A,c}$ have one of $[a_1,a_2]$, $[a_2,a_3]$ or $[a_3,a_4]$ as its optimal basis.

Given a polyhedron $P \subset \mathbb{R}^n$ and a face F of P, the *normal cone* of F at P is the cone $N_P(F) := \{\omega \in \mathbb{R}^n \,:\, \omega\cdot x' \geq \omega \cdot x, \text{ for all } x' \in F \text{ and } x \in P\}$. The normal cones of all faces of P form a cone complex in $\mathbb{R}^n$ called the *normal fan* of P.

Proposition 2.4. *The regular triangulation Δ_c of $cone(A)$ is the normal fan of the polyhedron $P_c := \{y \in \mathbb{R}^d \,:\, yA \leq c\}$.*

Proof. The polyhedron P_c is the feasible region of $maximize\ \{y \cdot b \,:\, yA \leq c,\ y \in \mathbb{R}^d\}$, the dual program to $LP_{A,c}(b)$. The support of the normal fan of P_c is $cone(A)$, since this is the polar cone of the recession cone $\{y \in \mathbb{R}^d : yA \leq 0\}$ of P_c. Suppose b is any vector in the interior of a maximal face $cone(A_\sigma)$ of Δ_c. Then by Proposition 2.3, $LP_{A,c}(b)$ has an optimal solution x^* with support σ. By complementary slackness, the optimal solution y to the dual of $LP_{A,c}(b)$ satisfies $y \cdot a_j = c_j$ for all $j \in \sigma$ and $y \cdot a_j \leq c_j$ otherwise. Since σ is a maximal face of Δ_c, $y \cdot a_j < c_j$ for all $j \notin \sigma$. Thus y is unique, and $cone(A_\sigma)$ is contained in the normal cone of P_c at the vertex y. If b lies in the interior of another maximal face $cone(A_\tau)$ then y', (the dual optimal solution to $LP_{A,c}(b)$) satisfies $y' \cdot A_\tau = c_\tau$ and $y' \cdot A_{\bar\tau} < c_{\bar\tau}$ where $\tau \neq \sigma$. As a result, y' is distinct from y, and each maximal cone in Δ_c lies in a distinct maximal cone in the normal fan of P_c. Since Δ_c and the normal fan of P_c are both cone complexes with the same support, they must therefore coincide. $\square$

Example 2.2 continued.
Figure 3 (a) shows the polyhedron P_c for Example 2.2 (i) with all its normal cones. The

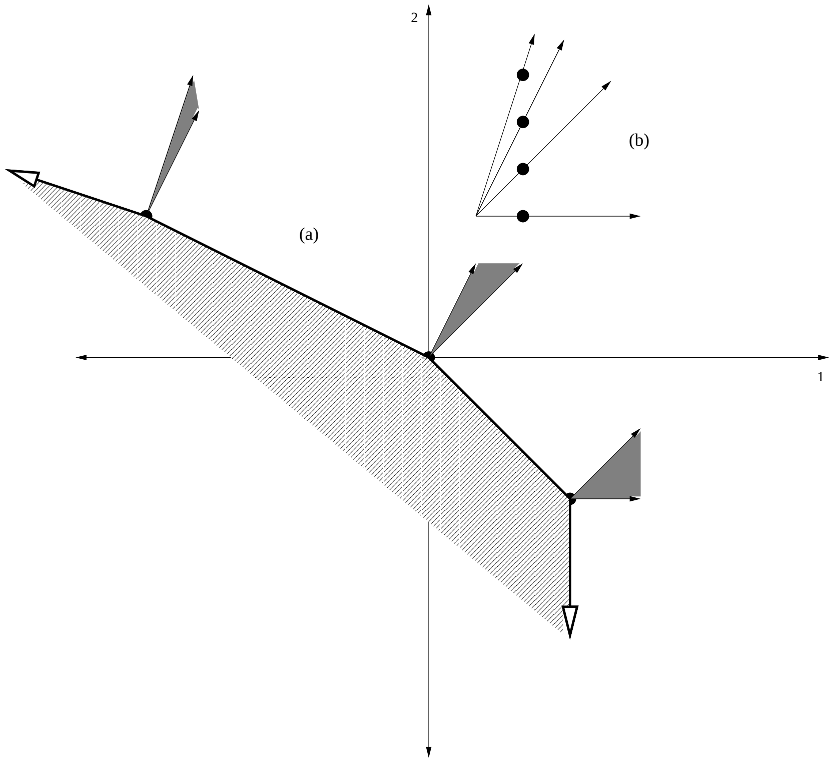

FIGURE 3. The polyhedron P_c and its normal fan for Example 2.2 (i).

normal fan of P_c is drawn in Figure 3 (b). Compare this fan with that in Figure 1 (b). $\square$

Corollary 2.5. *The polyhedron P_c is simple if and only if the regular subdivision Δ_c is a triangulation of cone(A).*

Regular triangulations were introduced by Gel'fand, Kapranov and Zelevinsky [14] and have various applications. They have played a central role in the algebraic study of integer programming ([39], [40]), and we use them now to define *group relaxations* of $IP_{A,c}(b)$.

A subset τ of $\{1,\ldots,n\}$ partitions $x=(x_1,\ldots,x_n)$ as x_τ and $x_{\bar\tau}$ where x_τ consists of the variables indexed by τ and $x_{\bar\tau}$ the variables indexed by the complementary set $\bar\tau$. Similarly, the matrix A is partitioned as $A=[A_\tau,A_{\bar\tau}]$ and the cost vector as $c=(c_\tau,c_{\bar\tau})$. If σ is a maximal face of Δ_c, then A_σ is nonsingular and $Ax=b$ can be written as $x_\sigma = A_\sigma^{-1}(b-A_{\bar\sigma}x_{\bar\sigma})$. Then $c\cdot x = c_\sigma(A_\sigma^{-1}(b-A_{\bar\sigma}x_{\bar\sigma}))+c_{\bar\sigma}x_{\bar\sigma} = c_\sigma A_\sigma^{-1}b+(c_{\bar\sigma}-c_\sigma A_\sigma^{-1}A_{\bar\sigma})x_{\bar\sigma}$. Let $\tilde c_{\bar\sigma} := c_{\bar\sigma}-c_\sigma A_\sigma^{-1}A_{\bar\sigma}$ and, for any face τ of σ, let $\tilde c_{\bar\tau}$ be the extension of $\tilde c_{\bar\sigma}$ to a vector in $\mathbb{R}^{|\bar\tau|}$ by adding zeros.

We now define a group relaxation of $IP_{A,c}(b)$ with respect to each face τ of Δ_c.

Definition 2.6. The group relaxation of the integer program $IP_{A,c}(b)$ with respect to the face τ of Δ_c is the program:

$$G^\tau(b) = minimize\ \{\tilde c_{\bar\tau}\cdot x_{\bar\tau}\ : A_\tau x_\tau + A_{\bar\tau}x_{\bar\tau} = b,\ x_{\bar\tau}\geq 0, (x_\tau, x_{\bar\tau})\in\mathbb{Z}^n\}.$$

Equivalently, $G^\tau(b) = minimize\ \{\tilde c_{\bar\tau}\cdot x_{\bar\tau}\ :\ A_{\bar\tau}x_{\bar\tau}\equiv b\ (mod\ \mathbb{Z}A_\tau),\ x_{\bar\tau}\geq 0,\ integer\}$ where $\mathbb{Z}A_\tau$ is the lattice generated by the columns of A_τ. Suppose $x^*_{\bar\tau}$ is an optimal solution to the latter formulation. Since τ is a face of Δ_c, the columns of A_τ are linearly independent, and therefore the linear system $A_\tau x_\tau + A_{\bar\tau} x^*_{\bar\tau} = b$ has a unique solution. Solving this system for x_τ, the optimal solution $x^*_{\bar\tau}$ of $G^\tau(b)$ can be uniquely lifted to the solution $(x^*_\tau, x^*_{\bar\tau})$ of $Ax = b$. The formulation of $G^\tau(b)$ in Definition 2.6 shows that x^*_τ is an integer vector. The group relaxation $G^\tau(b)$ solves $IP_{A,c}(b)$ if and only if x^*_τ is also non-negative.

The group relaxations of $IP_{A,c}(b)$ from Definition 2.6 contain among them the classical group relaxations of $IP_{A,c}(b)$ found in the literature. The program $G^\sigma(b)$, where A_σ is the optimal basis of the linear relaxation $LP_{A,c}(b)$, is precisely Gomory's group relaxation of $IP_{A,c}(b)$ [15]. The set of relaxations $G^\tau(b)$ as τ varies among the subsets of this σ are the *extended* group relaxations of $IP_{A,c}(b)$ defined by Wolsey [45]. Since $\emptyset \in \Delta_c$, $G^\emptyset(b) = IP_{A,c}(b)$ is a group relaxation of $IP_{A,c}(b)$, and hence $IP_{A,c}(b)$ will certainly be solved by one of its extended group relaxations. However, it is possible to construct examples where a group relaxation $G^\tau(b)$ solves $IP_{A,c}(b)$, but $G^\tau(b)$ is neither Gomory's group relaxation of $IP_{A,c}(b)$ nor one of its nontrivial extended Wolsey relaxations (see Example 4.2). Thus, Definition 2.6 typically creates more group relaxations for each program in $IP_{A,c}$ than in the classical situation. This has the obvious advantage that it increases the chance that $IP_{A,c}(b)$ will be solved by some non-trivial relaxation, although one may have to keep track of many more relaxations for each program. In Theorem 2.8, we will prove that Definition 2.6 is the best possible in the sense that the relaxations of $IP_{A,c}(b)$ defined there are precisely all the bounded group relaxations of the program.

The goal in the rest of this section is to describe a useful reformulation of the group problem $G^\tau(b)$ which is needed in the rest of the paper and in the proof of Theorem 2.8. Given a sublattice Λ of $\mathbb{Z}^n$, a cost vector $w \in \mathbb{R}^n$ and a vector $v \in \mathbb{N}^n$, the *lattice program* defined by this data is

$$\textit{minimize } \{w \cdot x \ : \ x \equiv v \ (\textit{mod } \Lambda), \ x \in \mathbb{N}^n\}.$$

Let $\mathcal{L}$ denote the $(n-d)$-dimensional saturated lattice $\{x \in \mathbb{Z}^n : Ax = 0\} \subseteq \mathbb{Z}^n$ and u be a feasible solution of the integer program $IP_{A,c}(b)$. Since $IP_{A,c}(b) = \textit{minimize } \{c \cdot x : Ax = b\ (= Au),\ x \in \mathbb{N}^n\}$ can be rewritten as $\textit{minimize } \{c \cdot x \ : \ x - u \in \mathcal{L},\ x \in \mathbb{N}^n\}$, $IP_{A,c}(b)$ is equivalent to the lattice program

$$\textit{minimize } \{c \cdot x \ : \ x \equiv u \,(\textit{mod } \mathcal{L}),\ x \in \mathbb{N}^n\}.$$

For $\tau \in \Delta_c$, let π_τ be the projection map from $\mathbb{R}^n \to \mathbb{R}^{|\bar\tau|}$ that kills all coordinates indexed by τ. Then $\mathcal{L}_\tau := \pi_\tau(\mathcal{L})$ is a sublattice of $\mathbb{Z}^{|\bar\tau|}$ that is isomorphic to $\mathcal{L}$: Clearly, $\pi_\tau : \mathcal{L} \to \mathcal{L}_\tau$ is a surjection. If $\pi_\tau(v) = \pi_\tau(v')$ for $v, v' \in \mathcal{L}$, then $A_\tau v_\tau + A_{\bar\tau} v_{\bar\tau} = 0 = A_\tau v'_\tau + A_{\bar\tau} v'_{\bar\tau}$, implies that $A_\tau(v_\tau - v'_\tau) = 0$. Then $v_\tau = v'_\tau$ since the columns of A_τ are linearly independent. Using this fact, $G^\tau(b)$ can also be reformulated as a lattice program:

$$\begin{aligned} G^\tau(b) &= \textit{minimize } \{\tilde{c}_{\bar\tau} \cdot x_{\bar\tau} \ : A_\tau x_\tau + A_{\bar\tau} x_{\bar\tau} = b,\ x_{\bar\tau} \geq 0, (x_\tau, x_{\bar\tau}) \in \mathbb{Z}^n\} \\ &= \textit{minimize } \{\tilde{c}_{\bar\tau} \cdot x_{\bar\tau} \ : \ (x_\tau, x_{\bar\tau})^t - (u_\tau, u_{\bar\tau})^t \in \mathcal{L},\ x_{\bar\tau} \in \mathbb{N}^{|\bar\tau|}\} \\ &= \textit{minimize } \{\tilde{c}_{\bar\tau} \cdot x_{\bar\tau} \ : \ x_{\bar\tau} - u_{\bar\tau} \in \mathcal{L}_\tau,\ x_{\bar\tau} \in \mathbb{N}^{|\bar\tau|}\} \\ &= \textit{minimize } \{\tilde{c}_{\bar\tau} \cdot x_{\bar\tau} \ : \ x_{\bar\tau} \equiv \pi_\tau(u) \,(\textit{mod } \mathcal{L}_\tau),\ x_{\bar\tau} \in \mathbb{N}^{|\bar\tau|}\}. \end{aligned}$$

Lattice programs were shown to be solved by Gröbner bases in [42]. Theorem 5.3 in [42] gives a geometric interpretation of these Gröbner bases in terms of corner polyhedra.

This paper was the first to make a connection between the theory of group relaxations and commutative algebra (see [42, §6]). Special results are possible when the sublattice Λ is of finite index. In particular, the associated Gröbner bases are easier to compute.

Since the $(n-d)$-dimensional lattice $\mathcal{L} \subset \mathbb{Z}^n$ is isomorphic to $\mathcal{L}_\tau \subset \mathbb{Z}^{|\bar{\tau}|}$ for $\tau \in \Delta_c$, $\mathcal{L}_\tau$ is of finite index if and only if τ is a maximal face of Δ_c. Hence, by the last sentence of the previous paragraph, the group relaxations $G^\sigma(b)$ as σ varies over the maximal faces of Δ_c are the easiest to solve among all group relaxations of $IP_{A,c}(b)$. They contain among them Gomory's group relaxation of $IP_{A,c}$. We give these relaxations a collective name.

Definition 2.7. The group relaxations $G^\sigma(b)$ of $IP_{A,c}(b)$, as σ varies among the maximal faces of Δ_c, are called the *Gomory relaxations* of $IP_{A,c}(b)$.

It is useful to reformulate $G^\tau(b)$ once again as follows. Let $B \in \mathbb{Z}^{n\times(n-d)}$ be any matrix such that the columns of B generate the lattice $\mathcal{L}$, and let u be a feasible solution of $IP_{A,c}(b)$ as before. Then

$$\begin{aligned} IP_{A,c}(b) &= \textit{minimize } \{c\cdot x \ :\ x - u \in \mathcal{L},\ x \in \mathbb{N}^n\} \\ &= \textit{minimize } \{c\cdot x \ :\ x = u - Bz,\ x \geq 0,\ z \in \mathbb{Z}^{n-d}\}. \end{aligned}$$

The last problem is equivalent to *minimize* $\{c\cdot(u - Bz) \ :\ Bz \leq u,\ z \in \mathbb{Z}^{n-d}\}$ and, therefore $IP_{A,c}(b)$ is equivalent to the problem

$$\textit{minimize } \{(-cB)\cdot z \ :\ Bz \leq u,\ z \in \mathbb{Z}^{n-d}\}. \tag{4}$$

There is a bijection between the set of feasible solutions of (4) and the set of feasible solutions of $IP_{A,c}(b)$ via the isomorphism $z \mapsto u - Bz$. In particular, $0 \in \mathbb{R}^{n-d}$ is feasible for (4) and it is the pre-image of u under this map.

If $B^{\bar{\tau}}$ denotes the $|\bar{\tau}| \times (n-d)$ submatrix of B obtained by deleting the rows indexed by τ, then $\mathcal{L}_\tau = \pi_\tau(\mathcal{L}) = \{B^{\bar{\tau}}z \ :\ z \in \mathbb{Z}^{n-d}\}$. Using the same techniques as above, $G^\tau(b)$ can be reformulated as

$$\textit{minimize } \{(-\tilde{c}_{\bar{\tau}}B^{\bar{\tau}})\cdot z \ :\ B^{\bar{\tau}}z \leq \pi_\tau(u),\ z \in \mathbb{Z}^{n-d}\}.$$

Since $\tilde{c}_{\bar{\tau}} = \pi_\tau(c - c_\sigma A_\sigma^{-1}A)$ for any maximal face σ of Δ_c containing τ and the support of $c - c_\sigma A_\sigma^{-1}A$ is contained in $\bar{\tau}$, $\tilde{c}_{\bar{\tau}}B^{\bar{\tau}} = (c - c_\sigma A_\sigma^{-1}A)B = cB$ since $AB = 0$. Hence $G^\tau(b)$ is equivalent to

$$\textit{minimize } \{(-cB)\cdot z \ :\ B^{\bar{\tau}}z \leq \pi_\tau(u),\ z \in \mathbb{Z}^{n-d}\}. \tag{5}$$

The feasible solutions to (4) are the lattice points in the rational polyhedron $P_u := \{z \in \mathbb{R}^{n-d} : Bz \leq u\}$, and the feasible solutions to (5) are the lattice points in the relaxation $P_u^{\bar{\tau}} := \{z \in \mathbb{R}^{n-d} : B^{\bar{\tau}}z \leq \pi_\tau(u)\}$ of P_u obtained by deleting the inequalities indexed by τ. In theory, one could define group relaxations of $IP_{A,c}(b)$ with respect to any $\tau \subseteq \{1,\ldots,n\}$. The following theorem illustrates the completeness of Definition 2.6.

Theorem 2.8. *The group relaxation $G^\tau(b)$ of $IP_{A,c}(b)$ has a finite optimal solution if and only if $\tau \subseteq \{1,\ldots,n\}$ is a face of Δ_c.*

Proof. Since all data are integral it suffices to prove that the linear relaxation

$$\textit{minimize } \{(-cB)\cdot z \ :\ z \in P_u^{\bar{\tau}}\}$$

is bounded if and only if $\tau \in \Delta_c$.

If τ is a face of Δ_c then there exists $y \in \mathbb{R}^d$ such that $yA_\tau = c_\tau$ and $yA_{\bar{\tau}} < c_{\bar{\tau}}$. Using the fact that $A_\tau B^\tau + A_{\bar{\tau}} B^{\bar{\tau}} = 0$ we see that $cB = c_\tau B^\tau + c_{\bar{\tau}} B^{\bar{\tau}} = yA_\tau B^\tau + c_{\bar{\tau}} B^{\bar{\tau}} = y(-A_{\bar{\tau}} B^{\bar{\tau}}) + c_{\bar{\tau}} B^{\bar{\tau}} = (c_{\bar{\tau}} - yA_{\bar{\tau}})B^{\bar{\tau}}$. This implies that cB is a positive linear combination of the rows of $B^{\bar{\tau}}$ since $c_{\bar{\tau}} - yA_{\bar{\tau}} > 0$. Hence cB lies in the polar of $\{z \in \mathbb{R}^{n-d} : B^{\bar{\tau}} z \leq 0\}$ which is the recession cone of $P_u^{\bar{\tau}}$ proving that the linear program *minimize* $\{(-cB) \cdot z \ : \ z \in P_u^{\bar{\tau}}\}$ is bounded.

The linear program *minimize* $\{(-cB) \cdot z \ : \ z \in P_u^{\bar{\tau}}\}$ is feasible since 0 is a feasible solution. If it is bounded as well then *minimize* $\{c_\tau x_\tau + c_{\bar{\tau}} x_{\bar{\tau}} \ : \ A_\tau x_\tau + A_{\bar{\tau}} x_{\bar{\tau}} = b, \, x_{\bar{\tau}} \geq 0\}$ is feasible and bounded. As a result, the dual of the latter program *maximize* $\{y \cdot b \ : \ yA_\tau = c_\tau, \, yA_{\bar{\tau}} \leq c_{\bar{\tau}}\}$ is feasible. This shows that a superset of τ is a face of Δ_c which implies that $\tau \in \Delta_c$ since Δ_c is a triangulation. □

3. Associated Sets

The group relaxation $G^\tau(b)$ (seen as (5)) solves the integer program $IP_{A,c}(b)$ (seen as (4)) if and only if both programs have the same optimal solution $z^* \in \mathbb{Z}^{n-d}$. If $G^\tau(b)$ solves $IP_{A,c}(b)$ then $G^{\tau'}(b)$ also solves $IP_{A,c}(b)$ for every $\tau' \subset \tau$ since $G^{\tau'}(b)$ is a *stricter* relaxation of $IP_{A,c}(b)$ (has more non-negativity restrictions) than $G^\tau(b)$. For the same reason, one would expect that $G^\tau(b)$ is easier to solve than $G^{\tau'}(b)$. Therefore, the most useful group relaxations of $IP_{A,c}(b)$ are those indexed by the *maximal* elements in the subcomplex of Δ_c consisting of all faces τ such that $G^\tau(b)$ solves $IP_{A,c}(b)$. The following definition isolates such relaxations.

Definition 3.1. A face τ of the regular triangulation Δ_c is an *associated set* of $IP_{A,c}$ (or is associated to $IP_{A,c}$) if for some $b \in \mathbb{N}A$, $G^\tau(b)$ solves $IP_{A,c}(b)$ but $G^{\tau'}(b)$ does not for all faces τ' of Δ_c such that $\tau \subset \tau'$.

The associated sets of $IP_{A,c}$ carry all the information about all the group relaxations needed to solve the programs in $IP_{A,c}$. In this section we will develop tools to understand these sets. We start by considering the set $\mathcal{O}_c \subset \mathbb{N}^n$ of all the optimal solutions of all programs in $IP_{A,c}$. A basic result in the algebraic study of integer programming is that $\mathcal{O}_c$ is an *order ideal* or *down set* in $\mathbb{N}^n$, i.e., if $u \in \mathcal{O}_c$ and $v \leq u, \ v \in \mathbb{N}^n$, then $v \in \mathcal{O}_c$. One way to prove this is to show that the complement $\mathcal{N}_c := \mathbb{N}^n \backslash \mathcal{O}_c$ has the property that if $v \in \mathcal{N}_c$ then $v + \mathbb{N}^n \subseteq \mathcal{N}_c$. Every lattice point in $\mathbb{N}^n$ is a feasible solution to a unique program in $IP_{A,c}$ ($u \in \mathbb{N}^n$ is feasible for $IP_{A,c}(Au)$). Hence, $\mathcal{N}_c$ is the set of all non-optimal solutions of all programs in $IP_{A,c}$. A set $P \subseteq \mathbb{N}^n$ with the property that $p + \mathbb{N}^n \subseteq P$ whenever $p \in P$ has a finite set of minimal elements. Hence there exists $\alpha_1, \ldots, \alpha_t \in \mathcal{N}_c$ such that

$$\mathcal{N}_c = \bigcup_{i=1}^{t} (\alpha_i + \mathbb{N}^n).$$

As a result, $\mathcal{O}_c$ is completely specified by the finitely many "generators" $\alpha_1, \ldots, \alpha_t$ of its complement $\mathcal{N}_c$. See [43] for proofs of these assertions. Recall that the cost vector c of $IP_{A,c}$ was assumed to be generic in the sense that each program in $IP_{A,c}$ has a unique optimal solution. This implies that there is a bijection between the lattice points of $\mathcal{O}_c$ and the

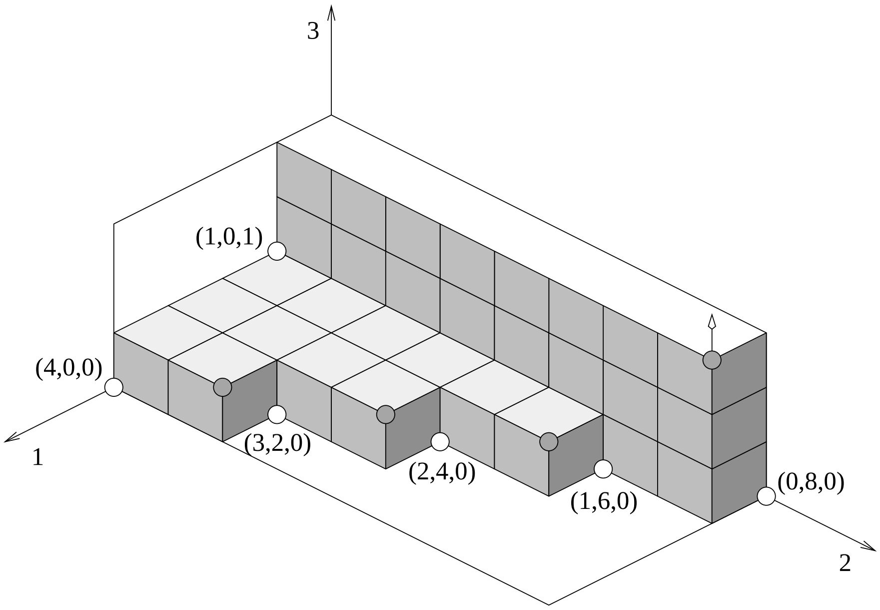

FIGURE 4. The set of non-optimal solutions $\mathcal{N}_c$ for Example 3.2.

semigroup $\mathbb{N}A$ via the map $\phi_A : \mathcal{O}_c \to \mathbb{N}A$ such that $u \mapsto Au$. The inverse of ϕ_A sends a vector $b \in \mathbb{N}A$ to the optimal solution of $IP_{A,c}(b)$.

Example 3.2. Consider the family of knapsack problems:

$$\textit{minimize}\ \{10000x_1 + 100x_2 + x_3 \ : \ 2x_1 + 5x_2 + 8x_3 = b,\ (x_1, x_2, x_3) \in \mathbb{N}^3\}$$

as b varies in the semigroup $\mathbb{N}[2\ 5\ 8]$. The set $\mathcal{N}_c$ is generated by the vectors

$$(0,8,0), (1,0,1), (1,6,0), (2,4,0), (3,2,0),\ \textit{and}\ (4,0,0)$$

which means that $\mathcal{N}_c = ((0,8,0) + \mathbb{N}^3) \cup \cdots \cup ((4,0,0) + \mathbb{N}^3)$. Figure 4 is a picture of $\mathcal{N}_c$ (created by Ezra Miller). The white points are its generators. One can see that $\mathcal{O}_c$ consists of finitely many points of the form $(p, q, 0)$ where $p \geq 1$ and the eight "lattice lines" of points $(0, i, *)$, $i = 0, \ldots, 7$. □

The most fundamental open question concerning $\mathcal{O}_c$ is the following.

Problem 3.3. *Characterize the order ideals in $\mathbb{N}^n$ that arise as $\mathcal{O}_c$ for a family of integer programs $IP_{A,c}$ where $A \in \mathbb{Z}^{d\times n}$ and $c \in \mathbb{Z}^n$ is generic.*

Several necessary conditions for an order ideal to be $\mathcal{O}_c$ are known, of which the *chain property* explained in Section 5 is the most sophisticated thus far. For the purpose of computations, it is most effective (as of now) to think of $\mathcal{N}_c$ and $\mathcal{O}_c$ algebraically[1]. These sets carry all of the information concerning the family $IP_{A,c}$ — the minimal test set (Gröbner basis) of the family, complete information on the group relaxations needed to solve all programs in the family, and precise sensitivity information for $IP_{A,c}$ to variations in the cost function c. The Gröbner bases approach to integer programming allows $\mathcal{N}_c$ (and thus $\mathcal{O}_c$) to be calculated

[1]See [**A1**] in Section 8.

via the Buchberger algorithm for Gröbner bases. Besides this, $\mathcal{O}_c$ can also be constructed by repeated calls to an integer programming oracle [26]. This second method is yet to be implemented and tested seriously. The following problem remains important.

Problem 3.4. *Find ways to construct and store $\mathcal{O}_c$ efficiently.*

We will now describe a certain decomposition of the set $\mathcal{O}_c$ which in turn will shed light on the associated sets of $IP_{A,c}$. For $u \in \mathbb{N}^n$, consider $Q_u := \{z \in \mathbb{R}^{n-d} \ : \ Bz \leq u, \ (-cB) \cdot z \leq 0\}$ and its relaxation $Q_u^{\bar{\tau}} := \{z \in \mathbb{R}^{n-d} \ : \ B^{\bar{\tau}} z \leq \pi_\tau(u), \ (-cB) \cdot z \leq 0\}$ where $B, B^{\bar{\tau}}$ are as in (4) and (5) and $\tau \in \Delta_c$. By Theorem 2.8, both Q_u and $Q_u^{\bar{\tau}}$ are polytopes. Notice that if $\pi_\tau(u) = \pi_\tau(u')$ for two distinct vectors $u, u' \in \mathbb{N}^n$, then $Q_u^{\bar{\tau}} = Q_{u'}^{\bar{\tau}}$.

Lemma 3.5. *(i) A lattice point u is in $\mathcal{O}_c$ if and only if $Q_u \cap \mathbb{Z}^{n-d} = \{0\}$.*
(ii) If $u \in \mathcal{O}_c$, then the group relaxation $G^\tau(Au)$ solves the integer program $IP_{A,c}(Au)$ if and only if $Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = \{0\}$.

Proof. (i) The lattice point u belongs to $\mathcal{O}_c$ if and only if u is the optimal solution to $IP_{A,c}(Au)$ which is equivalent to $0 \in \mathbb{Z}^{n-d}$ being the optimal solution to the reformulation (4) of $IP_{A,c}(Au)$. Since c is generic, the last statement is equivalent to $Q_u \cap \mathbb{Z}^{n-d} = \{0\}$. The second statement follows from (i) and the fact that (5) solves (4) if and only if they have the same optimal solution. □

In order to state the coming results, it is convenient to assume that the vector u in (4) and (5) is the optimal solution to $IP_{A,c}(b)$. For an element $u \in \mathcal{O}_c$ and a face τ of Δ_c let $S(u, \tau)$ be the *affine* semigroup $u + \mathbb{N}(e_i : i \in \tau) \subseteq \mathbb{N}^n$ where e_i denotes the ith unit vector of $\mathbb{R}^n$. Note that $S(u, \tau)$ is not a semigroup if $u \neq 0$, but is a translation of the semigroup $\mathbb{N}(e_i : i \in \tau)$. We use the adjective *affine* here as in an *affine subspace* which is not a subspace but the translation of one. Note that if $v \in S(u, \tau)$, then $\pi_\tau(v) = \pi_\tau(u)$.

Lemma 3.6. *For $u \in \mathcal{O}_c$ and a face τ of Δ_c, the affine semigroup $S(u, \tau)$ is contained in $\mathcal{O}_c$ if and only if $G^\tau(Au)$ solves $IP_{A,c}(Au)$.*

Proof. Suppose $S(u, \tau) \subseteq \mathcal{O}_c$. Then by Lemma 3.5 (i), for all $v \in S(u, \tau)$,

$$Q_v = \{z \in \mathbb{R}^{n-d} \ : \ B^\tau z \leq \pi_{\bar{\tau}}(v), \ B^{\bar{\tau}} z \leq \pi_\tau(u), \ (-cB) \cdot z \leq 0\} \cap \mathbb{Z}^{n-d} = \{0\}.$$

Since $\pi_{\bar{\tau}}(v)$ can be any vector in $\mathbb{N}^{|\tau|}$, $Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = \{0\}$. Hence, by Lemma 3.5 (ii), $G^\tau(Au)$ solves $IP_{A,c}(Au)$.

If $v \in S(u, \tau)$, then $\pi_\tau(u) = \pi_\tau(v)$, and hence $Q_u^{\bar{\tau}} = Q_v^{\bar{\tau}}$. Therefore, if $G^\tau(Au)$ solves $IP_{A,c}(Au)$, then $\{0\} = Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = Q_v^{\bar{\tau}} \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. Since $Q_v^{\bar{\tau}}$ is a relaxation of Q_v, $Q_v \cap \mathbb{Z}^{n-d} = \{0\}$ for all $v \in S(u, \tau)$ and hence by Lemma 3.5 (i), $S(u, \tau) \subseteq \mathcal{O}_c$. □

Lemma 3.7. *For $u \in \mathcal{O}_c$ and a face τ of Δ_c, $G^\tau(Au)$ solves $IP_{A,c}(Au)$ if and only if $G^\tau(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u, \tau)$.*

Proof. If $v \in S(u, \tau)$ and $G^\tau(Au)$ solves $IP_{A,c}(Au)$, then as seen before, $\{0\} = Q_u^{\bar{\tau}} \cap \mathbb{Z}^{n-d} = Q_v^{\bar{\tau}} \cap \mathbb{Z}^{n-d}$ for all $v \in S(u, \tau)$. By Lemma 3.5 (ii), $G^\tau(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u, \tau)$. The converse holds for the trivial reason that $u \in S(u, \tau)$. □

Corollary 3.8. *For $u \in \mathcal{O}_c$ and a face τ of Δ_c, the affine semigroup $S(u, \tau)$ is contained in $\mathcal{O}_c$ if and only if $G^\tau(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u, \tau)$.*

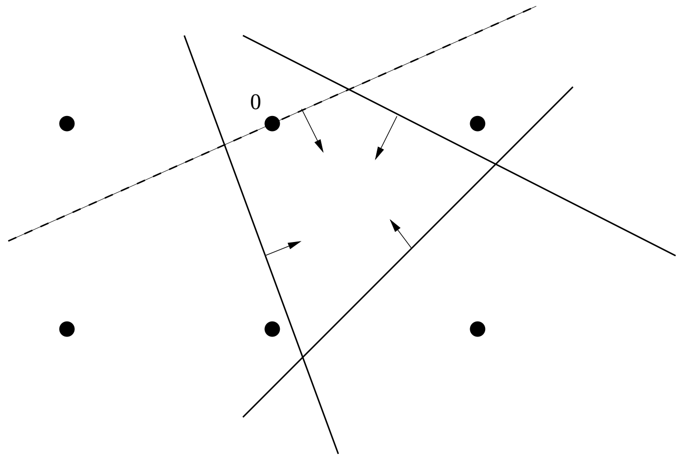

FIGURE 5. A standard polytope.

Since $\pi_\tau(u)$ determines the polytope $Q_u^{\bar\tau} = Q_v^{\bar\tau}$ for all $v \in S(u,\tau)$, we could have assumed that $supp(u) \subseteq \bar\tau$ in Lemmas 3.6 and 3.7.

Definition 3.9. For $\tau \in \Delta_c$ and $u \in \mathcal{O}_c$, (u,τ) is called an *admissible pair* of $\mathcal{O}_c$ if
(i) the support of u is contained in $\bar\tau$, and
(ii) $S(u,\tau) \subseteq \mathcal{O}_c$ or equivalently, $G^\tau(Av)$ solves $IP_{A,c}(Av)$ for all $v \in S(u,\tau)$.
An admissible pair (u,τ) is a *standard pair* of $\mathcal{O}_c$ if the affine semigroup $S(u,\tau)$ is not properly contained in $S(v,\tau')$ where (v,τ') is another admissible pair of $\mathcal{O}_c$.

Example 3.2 continued. From Figure 4, one can see that the standard pairs of $\mathcal{O}_c$ are

$$\begin{array}{ll} ((1,0,0),\emptyset) & ((1,3,0),\emptyset) \\ ((2,0,0),\emptyset) & ((2,3,0),\emptyset) \\ ((3,0,0),\emptyset) & ((1,4,0),\emptyset) \\ ((1,1,0),\emptyset) & ((1,5,0),\emptyset) \\ ((2,1,0),\emptyset) & \\ ((3,1,0),\emptyset) & \\ ((1,2,0),\emptyset) & \\ ((2,2,0),\emptyset) & \end{array} \quad \text{and} \quad \begin{array}{l} ((0,0,0),\{3\}) \\ ((0,1,0),\{3\}) \\ ((0,2,0),\{3\}) \\ ((0,3,0),\{3\}) \\ ((0,4,0),\{3\}) \\ ((0,5,0),\{3\}) \\ ((0,6,0),\{3\}) \\ ((0,7,0),\{3\}) \end{array}$$

□

Definition 3.10. For a face τ of Δ_c and a lattice point $u \in \mathbb{N}^n$, we say that the polytope $Q_u^{\bar\tau}$ is a *standard polytope* of $IP_{A,c}$ if $Q_u^{\bar\tau} \cap \mathbb{Z}^{n-d} = \{0\}$ and every relaxation of $Q_u^{\bar\tau}$ obtained by removing an inequality in $B^{\bar\tau} z \leq \pi_\tau(u)$ contains a non-zero lattice point.

Figure 5 is a diagram of a standard polytope $Q_u^{\bar\tau}$. The dashed line is the boundary of the half space $(-cB) \cdot z \leq 0$ while the other lines are the boundaries of the halfspaces given by the inequalities in $B^{\bar\tau} z \leq \pi_\tau(u)$. The origin is the only lattice point in the polytope and if any inequality in $B^{\bar\tau} z \leq \pi_\tau(u)$ is removed, a lattice point will enter the relaxation.

We re-emphasize that if $Q_u^{\bar\tau}$ is a standard polytope, then $Q_{u'}^{\bar\tau}$ is the same standard polytope if $\pi_\tau(u) = \pi_\tau(u')$. Hence the same standard polytope can be indexed by infinitely many

$u \in \mathbb{N}^n$. We now state the main result of this section which characterizes associated sets in terms of standard pairs and standard polytopes.

Theorem 3.11. *The following statements are equivalent:*
(i) The admissible pair (u, τ) is a standard pair of $\mathcal{O}_c$.
(ii) The polytope $Q_u^{\bar{\tau}}$ is a standard polytope of $IP_{A,c}$.
(iii) The face τ of Δ_c is associated to $IP_{A,c}$.

Proof. $(i) \Leftrightarrow (ii)$: The admissible pair (u, τ) is standard if and only if for every $i \in \bar{\tau}$, there exists some positive integer m_i and a vector $v \in S(u, \tau)$ such that $v + m_i e_i \in \mathcal{N}_c$. (If this condition did not hold for some $i \in \bar{\tau}$, then $(u', \tau \cup \{i\})$ would be an admissible pair of $\mathcal{O}_c$ such that $S(u', \tau \cup \{i\})$ contains $S(u, \tau)$ where u' is obtained from u by setting the ith component of u to zero. Conversely, if the condition holds for an admissible pair then the pair is standard.) Equivalently, for each $i \in \bar{\tau}$, there exists a positive integer m_i and a $v \in S(u, \tau)$ such that $Q^{\bar{\tau}}_{v+m_i e_i} = Q^{\bar{\tau}}_{u+m_i e_i}$ contains at least two lattice points. In other words, the removal of the inequality indexed by i from the inequalities in $B^{\bar{\tau}} z \leq \pi_\tau(u)$ will bring an extra lattice point into the corresponding relaxation of $Q_u^{\bar{\tau}}$. This is equivalent to saying that $Q_u^{\bar{\tau}}$ is a standard polytope of $IP_{A,c}$.

$(i) \Leftrightarrow (iii)$: Suppose (u, τ) is a standard pair of $\mathcal{O}_c$. Then $S(u, \tau) \subseteq \mathcal{O}_c$ and $G^\tau(Au)$ solves $IP_{A,c}(Au)$ by Lemma 3.6. Suppose $G^{\tau'}(Au)$ solves $IP_{A,c}(Au)$ for some face $\tau' \in \Delta_c$ such that $\tau \subset \tau'$. Lemma 3.6 then implies that $S(u, \tau')$ lies in $\mathcal{O}_c$. This contradicts the fact that (u, τ) was a standard pair of $\mathcal{O}_c$ since $S(u, \tau)$ is properly contained in $S(\hat{u}, \tau')$ corresponding to the admissible pair $(\hat{u}, \tau')$ where $\hat{u}$ is obtained from u by setting $u_i = 0$ for all $i \in \tau' \backslash \tau$.

To prove the converse, suppose τ is associated to $IP_{A,c}$. Then there exists some $b \in \mathbb{N}A$ such that $G^\tau(b)$ solves $IP_{A,c}(b)$ but $G^{\tau'}(b)$ does not for all faces τ' of Δ_c containing τ. Let u be the unique optimal solution of $IP_{A,c}(b)$. By Lemma 3.6, $S(u, \tau) \subseteq \mathcal{O}_c$. Let $\hat{u} \in \mathbb{N}^n$ be obtained from u by setting $u_i = 0$ for all $i \in \tau$. Then $G^\tau(A\hat{u})$ solves $IP_{A,c}(A\hat{u})$ since $Q_u^{\bar{\tau}} = Q_{\hat{u}}^{\bar{\tau}}$. Hence $S(\hat{u}, \tau) \subseteq \mathcal{O}_c$ and $(\hat{u}, \tau)$ is an admissible pair of $\mathcal{O}_c$. Suppose there exists another admissible pair (w, σ) such that $S(\hat{u}, \tau) \subset S(w, \sigma)$. Then $\tau \subseteq \sigma$. If $\tau = \sigma$ then $S(\hat{u}, \tau)$ and $S(w, \sigma)$ are both orthogonal translates of $\mathbb{N}(e_i \ : \ i \in \tau)$ and hence $S(\hat{u}, \tau)$ cannot be properly contained in $S(w, \sigma)$. Therefore, τ is a proper subset of σ which implies that $S(\hat{u}, \sigma) \subseteq \mathcal{O}_c$. Then, by Lemma 3.6, $G^\sigma(A\hat{u})$ solves $IP_{A,c}(A\hat{u})$ which contradicts that τ was an associated set of $IP_{A,c}$. $\square$

Example 3.2 continued. In Example 3.2 we can choose B to be the 3×2 matrix

$$B = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ -1 & -1 \end{bmatrix}.$$

The standard polytope defined by the standard pair $((1,0,0), \emptyset)$ is hence

$$\{(z_1, z_2) \in \mathbb{R}^2 \ : \ -z_1 + 4z_2 \leq 1, \ 2z_1 \leq 0, \ -z_1 - z_2 \leq 0, \ 9801z_1 - 40001z_2 \leq 0\}$$

while the standard polytope defined by the standard pair $((0,2,0), \{3\})$ is

$$\{(z_1, z_2) \in \mathbb{R}^2 \ : \ -z_1 + 4z_2 \leq 0, \ 2z_1 \leq 2, \ 9801z_1 - 40001z_2 \leq 0\}.$$

The associated sets of $IP_{A,c}$ in this example are $\emptyset$ and $\{3\}$. There are twelve quadrangular and eight triangular standard polytopes for this family of knapsack problems. □

Standard polytopes were introduced in [25], and the equivalence of parts (i) and (ii) of Theorem 3.11 was proved in [25, Theorem 2.5]. Under the linear map $\phi_A : \mathbb{N}^n \to \mathbb{N}A$ where $u \mapsto Au$, the affine semigroup $S(u, \tau)$ where (u, τ) is a standard pair of $\mathcal{O}_c$ maps to the affine semigroup $Au + \mathbb{N}A_\tau$ in $\mathbb{N}A$. Since every integer program in $IP_{A,c}$ is solved by one of its group relaxations, $\mathcal{O}_c$ is covered by the affine semigroups corresponding to its standard pairs. We call this cover and its image in $\mathbb{N}A$ under ϕ_A the *standard pair decompositions* of $\mathcal{O}_c$ and $\mathbb{N}A$, respectively. Since standard pairs of $\mathcal{O}_c$ are determined by the standard polytopes of $IP_{A,c}$, the standard pair decomposition of $\mathcal{O}_c$ is unique. The terminology used above has its origins in [41] which introduced the *standard pair decomposition* of a *monomial ideal.* The specialization to integer programming appear in [25], [26] and [39, §12.D]. The following theorem shows how the standard pair decomposition of $\mathcal{O}_c$ dictates which group relaxations solve which programs in $IP_{A,c}$.

Theorem 3.12. *Let v be the optimal solution of the integer program $IP_{A,c}(b)$. Then the group relaxation $G^\tau(Av)$ solves $IP_{A,c}(Av)$ if and only if there is some standard pair (u, τ') of $\mathcal{O}_c$ with $\tau \subseteq \tau'$ such that v belongs to the affine semigroup $S(u, \tau')$.*

Proof. Suppose v lies in $S(u, \tau')$ corresponding to the standard pair (u, τ') of $\mathcal{O}_c$. Then $S(v, \tau') \subseteq \mathcal{O}_c$ which implies that $G^{\tau'}(Av)$ solves $IP_{A,c}(Av)$ by Lemma 3.6. Hence $G^\tau(Av)$ also solves $IP_{A,c}(Av)$ for all $\tau \subseteq \tau'$.

To prove the converse, suppose τ' is a maximal element in the subcomplex of all faces τ of Δ_c such that $G^\tau(Av)$ solves $IP_{A,c}(Av)$. Then τ' is an associated set of $IP_{A,c}$. In the proof of $(iii) \Rightarrow (i)$ in Theorem 3.11, we showed that $(\hat{v}, \tau')$ is a standard pair of $\mathcal{O}_c$ where $\hat{v}$ is obtained from v by setting $v_i = 0$ for all $i \in \tau'$. Then $v \in S(\hat{v}, \tau')$. □

Example 3.2 continued. The eight standard pairs of $\mathcal{O}_c$ of the form $(*, \{3\})$, map to the eight affine semigroups:

$\mathbb{N}[8]$, $(5+\mathbb{N}[8])$, $(10+\mathbb{N}[8])$, $(15+\mathbb{N}[8])$, $(20+\mathbb{N}[8])$, $(25+\mathbb{N}[8])$, $(30+\mathbb{N}[8])$ *and* $(35+\mathbb{N}[8])$

contained in $\mathbb{N}A = \mathbb{N}[2, 5, 8] \subset \mathbb{N}$. For all right hand side vectors b in the union of these sets, the integer program $IP_{A,c}(b)$ can be solved by the group relaxation $G^{\{3\}}(b)$. The twelve standard pairs of the from $(*, \emptyset)$ map to the remaining finitely many points

$$2, 4, 6, 7, 9, 11, 12, 14, 17, 19, 22 \text{ and } 27$$

of $\mathbb{N}[2, 5, 8]$. If b is one of these points, then $IP_{A,c}(b)$ can only be solved as the full integer program. In this example, the regular triangulation $\Delta_c = \{\{3\}\}$. Hence $G^{\{3\}}(b)$ is a Gomory relaxation of $IP_{A,c}(b)$. □

For most $b \in \mathbb{N}A$, the program $IP_{A,c}(b)$ is solved by one of its Gomory relaxations, or equivalently, by Theorem 3.12, the optimal solution v of $IP_{A,c}(b)$ lies in $S(*, \sigma)$ for some standard pair $(*, \sigma)$ where σ is a maximal face of Δ_c. For mathematical versions of this informal statement, see [39, Proposition 12.16] and [15, Theorems 1 and 2]. Roughly speaking, these right hand sides are away from the boundary of $cone(A)$. (This was seen in Example 3.2

above, where for all but twelve right hand sides, $IP_{A,c}(b)$ was solvable by the Gomory relaxation $G^{\{3\}}(b)$. Further, these twelve right hand sides were toward the boundary of $cone(A)$, the origin in this one-dimensional case.) For the remaining right hand sides, $IP_{A,c}(b)$ can only be solved by $G^{\tau}(b)$ where τ is a lower dimensional face of Δ_c - possibly even the empty face. An important contribution of the approach described here is the identification of the minimal set of group relaxations needed to solve *all* programs in the family $IP_{A,c}$ and of the particular relaxations necessary to solve any given program in the family.

4. Arithmetic Degree

For an associated set τ of $IP_{A,c}$ there are only finitely many standard pairs of $\mathcal{O}_c$ indexed by τ since there are only finitely many standard polytopes of the form $Q_u^{\bar{\tau}}$. Borrowing terminology from [41], we call the number of standard pairs of the form $(\cdot, \tau)$ the *multiplicity* of τ in $\mathcal{O}_c$ (abbreviated as $mult(\tau)$). The total number of standard pairs of $\mathcal{O}_c$ is called the *arithmetic degree* of $\mathcal{O}_c$. Our main goal in this section is to provide bounds for these invariants of the the family $IP_{A,c}$ and discuss their relevance. We will need the following interpretation from Section 3.

Corollary 4.1. *The multiplicity of the face τ of Δ_c in $\mathcal{O}_c$ is the number of distinct standard polytopes of $IP_{A,c}$ indexed by $\bar{\tau}$, and the arithmetic degree of $\mathcal{O}_c$ is the total number of standard polytopes of $IP_{A,c}$.*

Proof. This result follows from Theorem 3.11. □

Example 3.2 continued. The multiplicity of the associated set $\{3\}$ is eight while the empty set has multiplicity twelve. The arithmetic degree of $\mathcal{O}_c$ is hence twenty. □

If the standard pair decomposition of $\mathcal{O}_c$ is known, then we can solve all programs in $IP_{A,c}$ by solving (arithmetic degree)-many *linear systems* as follows. For a given $b \in \mathbb{N}A$ and a standard pair (u, τ), consider the linear system

$$A_{\bar{\tau}}\pi_{\tau}(u) + A_{\tau}x = b, \text{ or equivalently, } A_{\tau}x = b - A_{\bar{\tau}}\pi_{\tau}(u). \tag{6}$$

As τ is a face of Δ_c, the columns of A_τ are linearly independent and the linear system (6) can be solved uniquely for x. Since the optimal solution of $IP_{A,c}(b)$ lies in $S(w, \sigma)$ for some standard pair (w, σ) of $\mathcal{O}_c$, at least one non-negative and integral solution for x will be found as we solve the linear systems (6) obtained by varying (u, τ) over all the standard pairs of $\mathcal{O}_c$. If the standard pair (u, τ) yields such a solution v, then $(\pi_\tau(u), v)$ is the optimal solution of $IP_{A,c}(b)$. This pre-processing of $IP_{A,c}$ has the same flavor as [30]. The main result in [30] is that given a coefficient matrix $A \in \mathbb{R}^{m\times n}$ and cost vector c, there exists *floor functions* $f_1, \ldots, f_k : \mathbb{R}^m \to \mathbb{Z}^n$ such that for a right hand side vector b, the optimal solution of the corresponding integer program is the one among $f_1(b), \ldots, f_k(b)$ that is feasible and attains the best objective function value. The crucial point is that this algorithm runs in time bounded above by a polynomial in the length of the data for fixed n and j, where j is the affine dimension of the space of right hand sides. In our situation, the preprocessing involves solving (arithmetic-degree)-many linear systems. Given this, it is interesting to bound the arithmetic degree of $\mathcal{O}_c$.

The second equation in (6) suggests that one could think of the first arguments u in the standard pairs (u, τ) of $\mathcal{O}_c$ as "correction vectors" that need to be applied to find the optimal solutions of programs in $IP_{A,c}$. Thus the arithmetic degree of $\mathcal{O}_c$ is the total number of correction vectors that are needed to solve all programs in $IP_{A,c}$. The multiplicities of associated sets give a finer count of these correction vectors, organized by faces of Δ_c. If the optimal solution of $IP_{A,c}(b)$ lies in the affine semigroup $S(w, \sigma)$ given by the standard pair (w, σ) of $\mathcal{O}_c$, then w is a correction vector for this b as well as all other b's in $(Aw + \mathbb{N}A_\sigma)$. One obtains all correction vectors for $IP_{A,c}$ by solving the (arithmetic degree)-many integer programs with right hand sides Au for all standard pairs (u, τ) of $\mathcal{O}_c$. See [47] for a similar result from the classical theory of group relaxations.

In Example 3.2, $\Delta_c = \{\{3\}\}$ and both its faces $\{3\}$ and $\emptyset$ are associated to $IP_{A,c}$. In general, not all faces of Δ_c need be associated sets of $IP_{A,c}$ and the poset of associated sets can be quite complicated. (We will study this poset in Section 5.) Hence, for $\tau \in \Delta_c$, $mult(\tau) = 0$ unless τ is an associated set of $IP_{A,c}$. We will now prove that all maximal faces of Δ_c are associated sets of $IP_{A,c}$. Further, if σ is a maximal face of Δ_c then $mult(\sigma)$ is the absolute value of $det(A_\sigma)$ divided by the g.c.d. of the maximal minors of A. This g.c.d is non-zero since A has full row rank. If the columns of A span an affine hyperplane, then the absolute value of $det(A_\sigma)$ divided by the g.c.d. of the maximal minors of A is called the *normalized volume* of the face σ in Δ_c. We first give a non-trivial example.

Example 4.2. Consider the rank three matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 2 & 1 & 0 \\ 0 & 5 & 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & 2 & 0 & 3 \end{bmatrix}$$

and the generic cost vector $c = (21, 6, 1, 0, 0, 0)$. The first three columns of A generate $cone(A)$ which is simplicial. The regular triangulation

$$\Delta_c = \{\{1, 3, 4\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}, \{4, 5, 6\}\}$$

is shown in Figure 6 as a triangulation of $conv(A)$. The six columns of A have been labeled by their column indices. The arithmetic degree of $\mathcal{O}_c$ in this example is 70. The following table shows all the standard pairs organized by associated sets and the multiplicity of each associated set. Note that all maximal faces of Δ_c are associated to $IP_{A,c}$. The g.c.d. of the maximal minors of A is five. Check that $mult(\sigma)$ is the normalized volume of σ whenever σ is a maximal face of Δ_c.

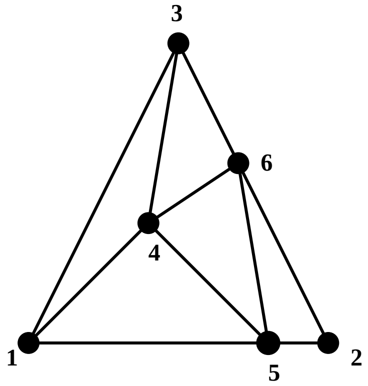

FIGURE 6. The regular triangulation Δ_c for Example 4.2.

τ	standard pairs $(\cdot,\tau)$	$mult(\tau)$
$\{1,3,4\}$	$(0,\cdot)$, $(e_5,\cdot)$, $(e_6,\cdot)$, $(e_5+e_6,\cdot)$, $(2e_6,\cdot)$	5
$\{1,4,5\}$	$(0,\cdot)$, $(e_2,\cdot)$, $(e_3,\cdot)$, $(e_6,\cdot)$, $(e_2+e_3,\cdot)$, $(2e_2,\cdot)$, $(3e_2,\cdot)$, $(2e_2+e_3,\cdot)$	8
$\{2,5,6\}$	$(0,\cdot)$, $(e_3,\cdot)$, $(2e_3,\cdot)$	3
$\{3,4,6\}$	$(0,\cdot)$, $(e_5,\cdot)$, $(2e_5,\cdot)$, $(3e_5,\cdot)$	4
$\{4,5,6\}$	$(0,\cdot)$, $(e_3,\cdot)$, $(2e_3,\cdot)$, $(3e_3,\cdot)$, $(4e_3,\cdot)$	5
$\{1,4\}$	$(e_3+2e_5+e_6,\cdot)$, $(2e_3+2e_5+e_6,\cdot)$, $(2e_3+2e_5,\cdot)$, $(2e_3+3e_5,\cdot)$, $(2e_3+4e_5,\cdot)$	5
$\{1,5\}$	$(e_2+e_6,\cdot)$, $(2e_2+e_6,\cdot)$, $(3e_2+e_6,\cdot)$	3
$\{2,5\}$	$(e_3+e_4,\cdot)$, $(e_4,\cdot)$, $(2e_4,\cdot)$	3
$\{3,4\}$	$(e_2,\cdot)$, $(e_1+e_2,\cdot)$, $(e_1+2e_5,\cdot)$, $(e_1+2e_5+e_6,\cdot)$, $(e_2+e_5,\cdot)$,	5
$\{3,6\}$	$(e_2,\cdot)$, $(e_2+e_5,\cdot)$	2
$\{4,5\}$	$(e_2+2e_3,\cdot)$, $(e_2+3e_3,\cdot)$, $(2e_2+2e_3,\cdot)$, $(3e_2+e_3,\cdot)$, $(4e_2,\cdot)$	5
$\{5,6\}$	$(e_2+3e_3,\cdot)$	1
$\{1\}$	$(e_2+e_3+e_6,\cdot)$, $(e_2+e_3+e_5+e_6,\cdot)$, $(e_2+2e_6,\cdot)$, $(e_2+e_3+2e_6,\cdot)$, $(2e_2+2e_6,\cdot)$, $(e_2+e_3+2e_5+e_6,\cdot)$	6
$\{3\}$	$(e_1+e_2+e_6,\cdot)$, $(e_1+e_2+2e_6,\cdot)$	2
$\{4\}$	$(e_1+e_2+2e_3+e_5,\cdot)$, $(e_1+e_2+2e_3+2e_5,\cdot)$, $(e_1+e_2+2e_3+3e_5,\cdot)$, $(e_1+e_2+2e_3+4e_5,\cdot)$, $(e_1+3e_3+3e_5,\cdot)$, $(e_1+3e_3+4e_5,\cdot)$	6
$\{\emptyset\}$	$(e_1+e_2+2e_3+e_5+e_6,\cdot)$, $(e_1+e_2+2e_3+2e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+e_6,\cdot)$, $(e_1+2e_2+e_3+e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+2e_5+e_6,\cdot)$, $(e_1+2e_2+e_3+2e_6,\cdot)$, $(e_1+3e_2+2e_6,\cdot)$	7
	Arithmetic Degree	70

Observe that the integer program $IP_{A,c}(b)$ where $b = A(e_1+e_2+e_3)$ is solved by $G^\tau(b)$ with $\tau = \{1,4,5\}$. By Proposition 2.3, Gomory's relaxation of $IP_{A,c}(b)$ is indexed by $\sigma = \{4,5,6\}$

since b lies in the interior of the face $cone(A_\sigma)$ of Δ_c. However, neither this relaxation nor any nontrivial extended relaxation solves $IP_{A,c}(b)$ since the optimal solution $e_1 + e_2 + e_3$ is not covered by any standard pair $(\cdot, \tau)$ where τ is a non-empty subset of $\{4, 5, 6\}$. □

Theorem 4.3. *For a set $\sigma \subseteq \{1, \dots, n\}$, $(0, \sigma)$ is a standard pair of $\mathcal{O}_c$ if and only if σ is a maximal face of Δ_c.*

Proof. If σ is a maximal face of Δ_c, then by Definition 2.1, there exists $y \in \mathbb{R}^d$ such that $yA_\sigma = c_\sigma$ and $yA_{\bar{\sigma}} < c_{\bar{\sigma}}$. Then $p = c_{\bar{\sigma}} - yA_{\bar{\sigma}} > 0$ and $pB^{\bar{\sigma}} = (c_{\bar{\sigma}} - yA_{\bar{\sigma}})B^{\bar{\sigma}} = c_{\bar{\sigma}}B^{\bar{\sigma}} - yA_{\bar{\sigma}}B^{\bar{\sigma}} = c_{\bar{\sigma}}B^{\bar{\sigma}} + yA_\sigma B^\sigma = c_{\bar{\sigma}}B^{\bar{\sigma}} + c_\sigma B^\sigma = cB$. Hence there is a positive dependence relation among $(-cB)$ and the rows of $B^{\bar{\sigma}}$. Since σ is a maximal face of Δ_c, $|det(A_\sigma)| \neq 0$. However, $|det(B^{\bar{\sigma}})| = |det(A_\sigma)|$ which implies that $|det(B^{\bar{\sigma}})| \neq 0$. Therefore, $(-cB)$ and the rows of $B^{\bar{\sigma}}$ span $\mathbb{R}^{n-d}$ positively. This implies that $Q_0^{\bar{\sigma}} = \{z \in \mathbb{R}^{n-d} \ : \ B^{\bar{\sigma}}z \leq 0, \ (-cB) \cdot z \leq 0\}$ is a polytope consisting of just the origin. If any inequality defining this simplex is dropped, the resulting relaxation is unbounded as only $n - d$ inequalities would remain. Hence $Q_0^{\bar{\sigma}}$ is a standard polytope of $IP_{A,c}$ and by Theorem 3.11, $(0, \sigma)$ is a standard pair of $\mathcal{O}_c$.

Conversely, if $(0, \sigma)$ is a standard pair of $\mathcal{O}_c$ then $Q_0^{\bar{\sigma}}$ is a standard polytope of $IP_{A,c}$. Since every inequality in the definition of $Q_0^{\bar{\sigma}}$ gives a halfspace containing the origin and $Q_0^{\bar{\sigma}}$ is a polytope, $Q_0^{\bar{\sigma}} = \{0\}$. Hence there is a positive linear dependence relation among $(-cB)$ and the rows of B^σ. If $|\bar{\sigma}| > n - d$, then $Q_0^{\bar{\sigma}}$ would coincide with the relaxation obtained by dropping some inequality from those in $B^{\bar{\sigma}}z \leq 0$. This would contradict that $Q_0^{\bar{\sigma}}$ was a standard polytope and hence $|\sigma| = d$ and σ is a maximal face of Δ_c. □

Corollary 4.4. *Every maximal face of Δ_c is an associated set of $IP_{A,c}$.*

For Theorem 4.5 and Corollary 4.6 below we assume that the g.c.d. of the maximal minors of A is one which implies that $\mathbb{Z}A = \mathbb{Z}^d$.

Theorem 4.5. *If σ is a maximal face of Δ_c then the multiplicity of σ in $\mathcal{O}_c$ is $|det(A_\sigma)|$.*

Proof. Consider the full dimensional lattice $\mathcal{L}_\sigma = \pi_\sigma(\mathcal{L}) = \{B^{\bar{\sigma}}z \ : \ z \in \mathbb{Z}^{n-d}\}$ in $\mathbb{Z}^{n-d}$. Since the g.c.d. of the maximal minors of A is assumed to be one, the lattice $\mathcal{L}_\sigma$ has index $|det(B^{\bar{\sigma}})| = |det(A_\sigma)|$ in $\mathbb{Z}^{n-d}$. Since $\mathcal{L}_\sigma$ is full dimensional, it has a strictly positive element which guarantees that each equivalence class of $\mathbb{Z}^{n-d}$ modulo $\mathcal{L}_\sigma$ has a non-negative member. This implies that there are $|det(A_\sigma)|$ distinct equivalence classes of $\mathbb{N}^{n-d}$ modulo $\mathcal{L}_\sigma$. Recall that if u is a feasible solution to $IP_{A,c}(b)$ then

$$G^\sigma(b) = minimize \ \{\tilde{c}_{\bar{\sigma}} \cdot x_{\bar{\sigma}} \ : \ x_{\bar{\sigma}} \equiv u_{\bar{\sigma}} \ (mod \ \mathcal{L}_\sigma), \ x_{\bar{\sigma}} \in \mathbb{N}^{n-d}\}.$$

Since there are $|det(A_\sigma)|$ equivalence classes of $\mathbb{N}^{n-d}$ modulo $\mathcal{L}_\sigma$, there are $|det(A_\sigma)|$ distinct group relaxations indexed by σ. The optimal solution of each program becomes the right hand side vector of a standard polytope (simplex) of $IP_{A,c}$ indexed by σ. Since no two optimal solutions are the same (as they come from different equivalence classes of $\mathbb{N}^{n-d}$ modulo $\mathcal{L}_\sigma$), there are precisely $|det(A_\sigma)|$ standard polytopes of $IP_{A,c}$ indexed by σ. □

Corollary 4.6. *The arithmetic degree of $\mathcal{O}_c$ is bounded below by the sum of the absolute values of $det(A_\sigma)$ as σ varies among the maximal faces of Δ_c.*

Theorem 4.5 gives a precise bound on the multiplicity of a maximal associated set of $IP_{A,c}$, which in turn provides a lower bound for the arithmetic degree of $\mathcal{O}_c$ in Corollary 4.6. No exact result like Theorem 4.5 is known when τ is a lower dimensional associated set of $IP_{A,c}$. Such bounds would provide a bound for the arithmetic degree of $\mathcal{O}_c$. The reader interested in the algebraic origins of some of the above results may consult the notes **[A2]** in Section 8. We close this section with a first attempt at bounding the arithmetic degree of $\mathcal{O}_c$ (under certain non-degeneracy assumptions). This result is due to Ravi Kannan, and its simple arguments are along the lines of proofs in [29] and [31].

Suppose $S \in \mathbb{Z}^{m \times n}$ and $u \in \mathbb{N}^m$ are fixed and $K_u := \{x \in \mathbb{R}^n \; : \; Sx \leq u\}$ is such that $K_u \cap \mathbb{Z}^n = \{0\}$ and the removal of any inequality defining K_u will bring in a non-zero lattice point into the relaxation. Let $s^{(i)}$ denote the ith row of S, $M := max||s^{(i)}||_1$ and $\Delta_k(S)$ and $\delta_k(S)$ be the maximum and minimum absolute values of the $k \times k$ subdeterminants of S. We will assume that $\delta_n(S) \neq 0$ which is a non-degeneracy condition on the data. We assume this set up in Theorem 4.8 and Lemmas 4.9 and 4.10.

Definition 4.7. If K is a convex set and v a non-zero vector in $\mathbb{R}^n$, the **width of K along** v, denoted as $width_v(K)$ is $max\ \{v \cdot x \; : \; x \in K\} - min\ \{v \cdot x \; : \; x \in K\}$.

Note that $width_v(K)$ is invariant under translations of K.

Theorem 4.8. *If K_u is as above then $0 \leq u_i \leq 2M(n+2)\frac{\Delta_n(S)}{\delta_n(S)}$.*

Lemma 4.9. *If K_u is as above then for some t, $1 \leq t \leq m$, $width_{s^{(t)}}(K_u) \leq M(n+2)$.*

Proof. Clearly, K_u is bounded since otherwise there would be a non-zero lattice point on an unbounded edge of K_u due to the integrality of all data. Suppose $width_{s^{(t)}}(K_u) > M(n+2)$ for all rows $s^{(t)}$ of S. Let p be the center of gravity of K_u. Then by a property of the center of gravity, for any $x \in K_u$, $(1/(n+1))$th of the vector from p to the reflection of x about p is also in K_u, i.e., $(1 + \frac{1}{n+1})p - \frac{1}{n+1}x \in K_u$. Fix i, $1 \leq i \leq m$ and let x_0 minimize $s^{(i)} \cdot x$ over K_u. By the definition of width, we then have $u_i - s^{(i)} \cdot x_0 > M(n+2)$ which implies that

$$s^{(i)} \cdot x_0 < u_i - M(n+2). \tag{7}$$

Now $s^{(i)} \cdot ((1 + \frac{1}{n+1})p - \frac{1}{n+1}x_0) \leq u_i$ implies that

$$s^{(i)} \cdot p \leq u_i(\frac{n+1}{n+2}) + \frac{s^{(i)} \cdot x_0}{n+2}. \tag{8}$$

Combining (7) and (8) we get

$$s^{(i)} \cdot p < u_i - M. \tag{9}$$

Let $q = \lfloor p \rfloor$ be the vector obtained by rounding down all components of p. Then $p = q + r$ where $0 \leq r_j < 1$ for all $j = 1, \ldots, n$, and by (9), $s^{(i)} \cdot (q + r) < u_i - M$ which leads to $s^{(i)} \cdot q + (s^{(i)} \cdot r + M) < u_i$. Since $M = max||s^{(i)}||_1$,

$$-M \leq s^{(i)} \cdot r \leq M. \tag{10}$$

and hence, $s^{(i)} \cdot q < u_i$. Repeating this argument for all rows of S, we get that $q \in K_u$. Similarly, if $q' = \lceil p \rceil$ is the vector obtained by rounding up all components of p, then $p = q' - r$ where $0 \leq r_j < 1$ for all $j = 1, \ldots, n$. Then (9) implies that $s^{(i)} \cdot (q' - r) < u_i - M$ which leads to $s^{(i)} \cdot q' + (M - s^{(i)} \cdot r) < u_i$. Again by (10), $s^{(i)} \cdot q' < u_i$ and hence $q' \in K_u$. Since $q \neq q'$, at least one of them is non-zero which contradicts that $K_u \cap \mathbb{Z}^n = \{0\}$. □

Lemma 4.10. *For any two rows $s^{(i)}, s^{(j)}$ of S, $width_{s^{(i)}}(K_u) \leq 2\frac{\Delta_n(S)}{\delta_n(S)} width_{s^{(j)}}(K_u)$.*

Proof. Without loss of generality we may assume that $j = n+1$. Since K_u is bounded, $width_{s^{(j)}}(K_u)$ is finite. Suppose the minimum of $s^{(j)} \cdot x$ over K_u is attained at v. Since translations leave the quantities in the lemma invariant, we may prove the lemma for the body $K_{u'}$ obtained by translating K_u by $-v$. Now $s^{(j)} \cdot x$ is minimized over $K_{u'}$ at the origin. By LP duality, there are n linearly independent constraints among the m defining $K_{u'}$ such that the minimum of $s^{(n+1)} \cdot x$ subject to just these n constraints is attained at 0. After renumbering the inequalities if necessary, assume these n constraints are the first n. Let

$$D = \{x \ : \ s^{(l)} \cdot x \leq u'_l, \ l = 1, 2, \ldots, n+1\}$$

where of course $u'_1 = u'_2 = \cdots = u'_n = 0$. Then by the above, D is a bounded simplex.

Since D contains $K_{u'}$, it suffices to show that for each i,

$$width_{s^{(i)}}(D) \leq 2(\frac{\Delta_n(S)}{\delta_n(S)}) width_{s^{(n+1)}}(K_{u'}) = 2(\frac{\Delta_n(S)}{\delta_n(S)}) u'_{n+1}. \tag{11}$$

We show that for each vertex q of D, $|s^{(i)} \cdot q| \leq (\frac{\Delta_n(S)}{\delta_n(S)}) u'_{n+1}$ which will prove (11). This is clearly true for $q = 0$. Without loss of generality assume that vertex q satisfies $s^{(l)} \cdot q = u'_l$ for $l = 2, 3, \ldots, n+1$. Since the determinant of the submatrix of S consisting of the rows $s^{(2)}, \ldots, s^{(n+1)}$ is not zero, for any i there exists rationals λ_l such that $s^{(i)} = \sum_{l=2}^{n+1} \lambda_l s^{(l)}$. By Cramer's rule, $|\lambda_l| \leq (\frac{\Delta_n(S)}{\delta_n(S)})$. Therefore, $s^{(i)} \cdot q = \sum_{l=2}^{n+1} \lambda_l s^{(l)} \cdot q = \sum_{l=2}^{n+1} \lambda_l u'_l = \lambda_{n+1} u'_{n+1}$ since $u'_l = 0$ for $l = 2, \ldots, n$. This proves that

$$|s^{(i)} \cdot q| = |\lambda_{n+1} u'_{n+1}| = |\lambda_{n+1}| u'_{n+1} \leq (\frac{\Delta_n(S)}{\delta_n(S)}) u'_{n+1}.$$

□

Proof of Theorem 4.8. From Lemmas 4.9 and 4.10 it follows that for any i, $1 \leq i \leq m$, $width_{s^{(i)}}(K_u) \leq 2(\frac{\Delta_n(S)}{\delta_n(S)}) M(n+2) = 2M(n+2)(\frac{\Delta_n(S)}{\delta_n(S)})$. Since $0 \in K_u$, $min\{s^{(i)} \cdot x \ : \ x \in K_u\} \leq 0$ while $max\{s^{(i)} \cdot x \ : \ x \in K_u\} = u_i$. Therefore, $u_i = u_i - 0 \leq width_{s^{(i)}}(K_u)$ and hence, $0 \leq u_i \leq 2M(n+2)(\frac{\Delta_n(S)}{\delta_n(S)})$ for all $1 \leq i \leq m$. □

Reverting back to our set up, let $\bar{B} = \begin{bmatrix} B \\ -cB \end{bmatrix}$. Suppose K_u is the standard polytope $Q_u^{\bar{\tau}}$. By Theorem 4.8, $0 \leq u_i \leq 2M(n-d+2)(\frac{\Delta_n(\bar{B})}{\delta_n(\bar{B})})$.

Corollary 4.11. *If no maximal minor of $\bar{B}$ is zero, then the arithmetic degree of $\mathcal{O}_c$ is at most $\left(2M(n-d+2)(\frac{\Delta_n(\bar{B})}{\delta_n(\bar{B})})\right)^n$.*

The above arguments do not use the condition that the removal of an inequality from K_u will bring in a lattice point into the relaxation. Further, the bound is independent of the number of facets of K_u, and Corollary 4.11 is straightforward. Thus, further improvements

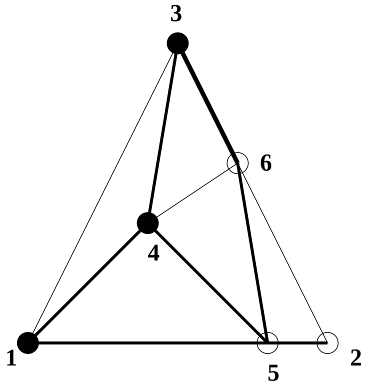

FIGURE 7. Lower dimensional associated sets of Example 4.2 except the empty set.

may be possible with more effort. However, these proofs provide a first bound for arithmetic degree and have the nice feature that they build a bridge to techniques from the geometry of numbers that have played a central role in theoretical integer programming in the work of Kannan, Lenstra, Lovász, Scarf and others. See [32] for a survey.

Problem 4.12. *Is it possible to find improved bounds for the multiplicities of associated sets and the arithmetic degree of $\mathcal{O}_c$?*

5. THE CHAIN THEOREM

We now examine the structure of the poset of associated sets of $IP_{A,c}$ which we denote as $Assets(IP_{A,c})$. All elements of $Assets(IP_{A,c})$ are faces of the regular triangulation Δ_c and the partial order is set inclusion. Theorem 4.3 provides a first result.

Corollary 5.1. *The maximal elements of $Assets(IP_{A,c})$ are the maximal faces of Δ_c.*

Example 4.2 continued. The lower dimensional associated sets of this example (except the empty set) are the thick faces of Δ_c shown in Figure 7. □

Despite the seemingly chaotic structure of $Assets(IP_{A,c})$ beyond its maximal elements, it has an important structural property that we now explain. (The proof of Theorem 5.2 relies on simple geometric ideas. Following the seemingly technical arguments with a picture could be very helpful. For algebraic comments on Theorem 5.2 see [**A3**] in Section 8.)

Theorem 5.2. **[The Chain Theorem]** *If $\tau \in \Delta_c$ is an associated set of $IP_{A,c}$ and $|\tau| < d$ then there exists a face $\tau' \in \Delta_c$ that is also an associated set of $IP_{A,c}$ with the property that $\tau \subset \tau'$ and $|\tau' \backslash \tau| = 1$.*

Proof. Since τ is an associated set of $IP_{A,c}$, by Theorem 3.11, $\mathcal{O}_c$ has a standard pair of the form (v, τ) and $Q_v^{\bar{\tau}} = \{z \in \mathbb{R}^{n-d} \,:\, B^{\bar{\tau}} z \leq \pi_\tau(v),\ (-cB) \cdot z \leq 0\}$ is a standard polytope of $IP_{A,c}$. Since $|\tau| < d$, τ is not a maximal face of Δ_c and hence by Theorem 4.3, $v \neq 0$. For each $i \in \bar{\tau}$, let R^i be the relaxation of $Q_v^{\bar{\tau}}$ obtained by removing the ith inequality $b_i \cdot z \leq v_i$ from $B^{\bar{\tau}} z \leq \pi_\tau(v)$, i.e.,

$$R^i := \{z \in \mathbb{R}^{n-d} \,:\, B^{\bar{\tau} \backslash \{i\}} z \leq \pi_{\tau \cup \{i\}}(v),\ (-cB) \cdot z \leq 0\}.$$

Let $E^i := R^i \backslash Q_v^{\bar{\tau}}$. Clearly, $E^i \cap Q_v^{\bar{\tau}} = \emptyset$, and, since the removal of $b_i \cdot z \leq v_i$ introduces at least one lattice point into R^i, $E^i \cap \mathbb{Z}^{n-d} \neq \emptyset$. Let z_i^* be the optimal solution to *minimize* $\{(-cB) \cdot z : z \in E^i \cap \mathbb{Z}^{n-d}\}$ if the program is bounded. This integer program is always feasible since $E^i \cap \mathbb{Z}^{n-d} \neq \emptyset$, but it may not have a finite optimal value. However, there exists at least one $i \in \bar{\tau}$ for which the above integer program is bounded. To see this, pick a maximal simplex $\sigma \in \Delta_c$ such that $\tau \subset \sigma$. The polytope $\{z \in \mathbb{R}^{n-d} : B^{\bar{\sigma}} z \leq \pi_\sigma(v),\ (-cB) \cdot z \leq 0\}$ is a simplex and hence bounded. This polytope contains all E^i for $i \in \sigma \backslash \tau$, and hence all these E^i are bounded and have finite optima with respect to $(-cB) \cdot z$. We may assume that the inequalities in $B^{\bar{\tau}} z \leq \pi_\tau(v)$ are labeled so that the finite optimal values are ordered as $(-cB) \cdot z_1^* \geq (-cB) \cdot z_2^* \geq \cdots \geq (-cB) \cdot z_p^*$ where $\{1, 2, \ldots, p\} \subseteq \bar{\tau}$.

Claim: *Let* $N^1 := \{z \in \mathbb{R}^{n-d} : B^{\bar{\tau} \backslash \{1\}} z \leq \pi_{\tau \cup \{1\}}(v),\ (-cB) \cdot z \leq (-cB) \cdot z_1^*\}$. *Then* z_1^* *is the unique lattice point in* N^1 *and the removal of any inequality from* $B^{\bar{\tau} \backslash \{1\}} z \leq \pi_{\tau \cup \{1\}}(v)$ *will bring in a new lattice point into the relaxation.*

Proof. Since z_1^* lies in R^1, $0 = (-cB) \cdot 0 \geq (-cB) \cdot z_1^*$. However, $0 > (-cB) \cdot z_1^*$ since otherwise, both z_1^* and 0 would be optimal solutions to *minimize*$\{(-cB) \cdot z : z \in R^1\}$ contradicting that c is generic. Therefore,

$$\begin{aligned} N^1 = &\ R^1 \cap \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z_1^*\} \\ = &\ (E^1 \cup Q_v^{\bar{\tau}}) \cap \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z_1^*\} \\ = &\ (E^1 \cap \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z_1^*\}) \\ &\ \bigcup (Q_v^{\bar{\tau}} \cap \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z_1^*\}). \end{aligned}$$

Since c is generic, z_1^* is the unique lattice point in the first polytope and the second polytope is free of lattice points. Hence z_1^* is the unique lattice point in N^1. The relaxation of N^1 got by removing $b_j \cdot z \leq v_j$ is the polyhedron $N^1 \cup (E^j \cap \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq (-cB) \cdot z_1^*\})$ for $j \in \bar{\tau}$ and $j \neq 1$. Either this is unbounded, in which case there is a lattice point z in this relaxation such that $(-cB) \cdot z_1^* \geq (-cB) \cdot z$, or (if $j \leq p$) we have $(-cB) \cdot z_1^* \geq (-cB) \cdot z_j^*$ and z_j^* lies in this relaxation. $\diamondsuit$

Translating N^1 by $-z_1^*$ we get $Q_{v'}^{\bar{\tau} \backslash \{1\}} := \{z \in \mathbb{R}^{n-d} : (-cB) \cdot z \leq 0,\ B^{\bar{\tau} \backslash \{1\}} z \leq v'\}$ where $v' = \pi_{\tau \cup \{1\}}(v) - B^{\bar{\tau} \backslash \{1\}} z_1^* \geq 0$ since z_1^* is feasible for all inequalities except the first one. Now $Q_{v'}^{\bar{\tau} \backslash \{1\}} \cap \mathbb{Z}^{n-d} = \{0\}$, and hence $(v', \tau \cup \{1\})$ is a standard pair of $\mathcal{O}_c$. $\square$

Example 4.2 continued. The empty set is associated to $IP_{A,c}$ and $\emptyset \subset \{1\} \subset \{1,4\} \subset \{1,4,5\}$ is a saturated chain in $Assets(IP_{A,c})$ that starts at the empty set. $\square$

Since the elements of $Assets(IP_{A,c})$ are faces of Δ_c, a maximal face of which is a d-element set, the length of a maximal chain in $Assets(IP_{A,c})$ is at most d. We denote the maximal length of a chain in $Assets(IP_{A,c})$ by $length(Assets(IP_{A,c}))$. When $n - d$ (the corank of A) is small compared to d, $length(Assets(IP_{A,c}))$ has a stronger upper bound than d. We use the following result of Bell and Scarf to prove the bound.

Theorem 5.3. [36, Corollary 16.5 a] *Let* $Ax \leq b$ *be a system of linear inequalities in* n *variables, and let* $c \in \mathbb{R}^n$. *If max* $\{c \cdot x : Ax \leq b,\ x \in \mathbb{Z}^n\}$ *is a finite number, then max* $\{c \cdot x : Ax \leq b,\ x \in \mathbb{Z}^n\} =$ *max* $\{c \cdot x : A'x \leq b', x \in \mathbb{Z}^n\}$ *for some subsystem* $A'x \leq b'$ *of* $Ax \leq b$ *with at most* $2^n - 1$ *inequalities.*

Theorem 5.4. *The length of a maximal chain in the poset of associated sets of $IP_{A,c}$ is at most $min(d, 2^{n-d} - (n-d+1))$.*

Proof. As seen earlier, $length(Assets(IP_{A,c})) \leq d$. If v lies in $\mathcal{O}_c$, then the origin is the optimal solution to the integer program $minimize\ \{(-cB) \cdot z\ :\ Bz \leq v,\ z \in \mathbb{Z}^{n-d}\}$. By Theorem 5.3, we need at most $2^{n-d} - 1$ inequalities to describe the same integer program which means that we can remove at least $n - (2^{n-d} - 1)$ inequalities from $Bz \leq v$ without changing the optimum. Assuming that the inequalities removed are indexed by τ, $Q_v^{\bar{\tau}}$ will be a standard polytope of $IP_{A,c}$. Therefore, $|\tau| \geq n - (2^{n-d} - 1)$. This implies that the maximal length of a chain in $Assets(IP_{A,c})$ is at most $d - (n - (2^{n-d} - 1)) = 2^{n-d} - (n - d + 1)$. □

Corollary 5.5. *The cardinality of an associated set of $IP_{A,c}$ is at least $max(0, n - (2^{n-d} - 1))$.*

Corollary 5.6. *If $n - d = 2$, then $length(Assets(IP_{A,c})) \leq 1$.*

Proof. In this situation, $2^{n-d} - (n - d + 1) = 4 - (4 - 2 + 1) = 4 - 3 = 1$. □

We conclude this section with a family of examples for which $length(Assets(IP_{A,c})) = 2^{n-d} - (n-d+1)$. This is adapted from [25, Proposition 3.9] which was modeled on a family of examples from [35].

Proposition 5.7. *For each $m > 1$, there is an integer matrix A of corank m and a cost vector $c \in \mathbb{Z}^n$ where $n = 2^m - 1$ such that $length(Assets(IP_{A,c})) = 2^m - (m + 1)$.*

Proof. Given $m > 1$, let $B' = (b_{ij}) \in \mathbb{Z}^{(2^m-1)\times m}$ be the matrix whose rows are all the $\{1, -1\}$-vectors in $\mathbf{R}^m$ except $v = (-1, -1, \ldots, -1)$. Let $B \in \mathbb{Z}^{(2^m+m-1)\times m}$ be obtained by stacking B' on top of $-I_m$ where I_m is the $m \times m$ identity matrix. Set $n = 2^m + m - 1$, $d = 2^m - 1$ and $A' = [I_d|B'] \in \mathbb{Z}^{d\times n}$. By construction, the columns of B span the lattice $\{u \in \mathbb{Z}^n\ :\ A'u = 0\}$. We may assume that the first row of B' is $(1, 1, \ldots, 1) \in \mathbb{R}^m$. Adding this row to all other rows of A' we get $A \in \mathbb{N}^{d\times n}$ with the same row space as A'. Hence the columns of B are also a basis for the lattice $\{u \in \mathbb{Z}^n\ :\ Au = 0\}$. Since the rows of B span $\mathbb{Z}^m$ as a lattice, we can find a cost vector $c \in \mathbb{Z}^n$ such that $(-cB) = v$.

For each row b_i of B' set $r_i := |\{b_{ij}\ :\ b_{ij} = 1\}|$, and let r be the vector of all r_is. By construction, the polytope $Q := \{z \in \mathbb{R}^m : B'z \leq r, -(cB) \cdot z \leq 0\}$ has no lattice points in its interior, and each of its 2^m facets has exactly one vertex of the unit cube in $\mathbb{R}^m$ in its relative interior. If we let $w_i = r_i - 1$, then the polytope $\{z \in \mathbb{R}^m : B'z \leq w, -(cB) \cdot z \leq 0\}$ is a standard polytope $Q_u^{\bar{\tau}}$ of $IP_{A,c}$ where $\tau = \{d + 1, d + 2, \ldots, d + m = n\}$ and $w = \pi_\tau(u)$. Since a maximal face of Δ_c is a $d = (2^m - 1)$-element set and $|\tau| = m$, Theorem 5.2 implies that $length(Assets(IP_{A,c})) \geq 2^m - 1 - m = 2^m - (m + 1)$. However, by Theorem 5.4, $length(Assets(IP_{A,c})) = min(2^m - 1, 2^m - (m + 1)) = 2^m - (m + 1)$ since $m > 1$ by assumption. □

Example 5.8. If we choose $m = 3$ then $n = 2^m + m - 1 = 10$ and $d = 2^m - 1 = 7$. Constructing B' and A as in Proposition 5.7, we get

$$B' = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

The vector $c = (11, 0, 0, 0, 0, 0, 0, 10, 10, 10)$ satisfies $(-cB) = (-1, -1, -1)$. The associated sets of $IP_{A,c}$ along with their multiplicities are given below.

τ	*Multiplicity*	τ	*Multiplicity*
$\{4,5,6,7,8,9,10\}^*$	4	$\{2,3,7,8,9,10\}$	2
$\{1,5,6,7,8,9,10\}$	4	$\{5,6,7,8,9,10\}^*$	1
$\{3,4,6,7,8,9,10\}$	4	$\{4,5,6,7,8,9\}$	1
$\{2,3,4,6,7,9,10\}$	2	$\{2,4,7,8,9,10\}$	2
$\{2,3,4,7,8,9,10\}$	4	$\{1,5,7,8,9,10\}$	1
$\{3,4,5,6,7,8,10\}$	2	$\{2,3,4,8,9,10\}$	1
$\{2,3,4,5,6,7,10\}$	1	$\{4,5,7,8,9,10\}$	2
$\{2,4,5,6,7,9,10\}$	2	$\{2,5,6,7,9,10\}$	1
$\{2,3,6,7,9,10\}$	1	$\{4,5,6,8,9,10\}$	2
$\{3,4,5,6,8,10\}$	1	$\{1,5,6,8,9,10\}$	1
$\{2,4,5,7,9,10\}$	1	$\{3,4,6,8,9,10\}$	2
$\{1,6,7,8,9,10\}$	1	$\{6,7,8,9,10\}^*$	1
$\{3,5,6,7,8,10\}$	1	$\{7,8,9,10\}^*$	1
$\{3,6,7,8,9,10\}$	2	$\{8,9,10\}^*$	1

The elements in the unique maximal chain in $Assets(IP_{A,c})$ are marked with a $*$ and $length(Assets(IP_{A,c})) = 2^3 - (3+1) = 4$ as predicted by Proposition 5.7. □

6. Gomory Integer Programs

Recall from Definition 2.7 that a group relaxation $G^\sigma(b)$ of $IP_{A,c}(b)$ is called a Gomory relaxation if σ is a maximal face of Δ_c. As discussed in Section 2, these relaxations are the easiest to solve among all relaxations of $IP_{A,c}(b)$. Hence it is natural to ask under what conditions on A and c would all programs in $IP_{A,c}$ be solvable by Gomory relaxations. We study this question in this section. The majority of the results here are taken from [24].

Definition 6.1. The family of integer programs $IP_{A,c}$ is a **Gomory family** if, for *every* $b \in \mathbb{N}A$, $IP_{A,c}(b)$ is solved by a group relaxation $G^\sigma(b)$ where σ is a *maximal* face of the regular triangulation Δ_c.

Theorem 6.2. *The following conditions are equivalent:*
(i) $IP_{A,c}$ *is a Gomory family.*
(ii) The associated sets of $IP_{A,c}$ *are precisely the maximal faces of* Δ_c.

(iii) $(*, \tau)$ *is a standard pair of* $\mathcal{O}_c$ *if and only if* τ *is a maximal face of* Δ_c.
(iv) All standard polytopes of $IP_{A,c}$ *are simplices.*

Proof. By Definition 6.1, $IP_{A,c}$ is a Gomory family if and only if for all $b \in \mathbb{N}A$, $IP_{A,c}(b)$ can be solved by one of its Gomory relaxations. By Theorem 3.12, this is equivalent to saying that every $u \in \mathcal{O}_c$ lies in some $S(*, \sigma)$ where σ is a maximal face of Δ_c and $(*, \sigma)$ a standard pair of $\mathcal{O}_c$. Definition 3.1 then implies that all associated sets of $IP_{A,c}$ are maximal faces of Δ_c. By Theorem 4.3, every maximal face of Δ_c is an associated set of $IP_{A,c}$ and hence $(i) \Leftrightarrow (ii)$. The equivalence of statements (ii), (iii) and (iv) follow from Theorem 3.11. □

If c is a generic cost vector such that for a triangulation Δ of $cone(A)$, $\Delta = \Delta_c$, then we say that Δ *supports* the order ideal $\mathcal{O}_c$ and the family of integer programs $IP_{A,c}$. No regular triangulation of the matrix A in Example 4.2 supports a Gomory family. Here is a matrix with a Gomory family.

Example 6.3. Consider the 3×6 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

In this case, $cone(A)$ has 14 distinct regular triangulations and 48 distinct sets $\mathcal{O}_c$ as c varies among all generic cost vectors. Ten of these triangulations support Gomory families; one for each triangulation. For instance, if $c = (0,\ 0,\ 1,\ 1,\ 0,\ 3)$, then

$$\Delta_c = \{\sigma_1 = \{1,2,5\},\ \sigma_2 = \{1,4,5\},\ \sigma_3 = \{2,5,6\},\ \sigma_4 = \{4,5,6\}\}$$

and $IP_{A,c}$ is a Gomory family since the standard pairs of $\mathcal{O}_c$ are:

$$(0, \sigma_1),\ (e_3, \sigma_1),\ (e_4, \sigma_1),\ (0, \sigma_2),\ (0, \sigma_3),\ \text{and}\ (0, \sigma_4).$$

□

The algebraic approach to integer programming allows one to compute all down sets $\mathcal{O}_c$ of a fixed matrix A as c varies among the set of generic cost vectors. See [27], [39] and [40] for details. The software package TiGERS [2] is custom-tailored for this purpose. The above example as well as many of the remaining examples in the paper were done using TiGERS. See [**A4**] in Section 8 for comments on the algebraic equivalent of a Gomory family.

We now compare the notion of a Gomory family to the classical notion of *total dual integrality* [36, §22]. It will be convenient to assume that $\mathbb{Z}A = \mathbb{Z}^d$ for these results.

Definition 6.4. The system $yA \leq c$ is **totally dual integral (TDI)** if $LP_{A,c}(b)$ has an integral optimal solution for each $b \in cone(A) \cap \mathbb{Z}^d$.

Definition 6.5. The regular triangulation Δ_c is *unimodular* if $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every maximal face $\sigma \in \Delta_c$.

Example 6.6. The regular triangulation in Example 2.2 (i) is unimodular while those in Example 2.2 (ii) and (iii) are not. □

Lemma 6.7. *The system $yA \leq c$ is TDI if and only if the regular triangulation Δ_c is unimodular.*

Proof. The regular triangulation Δ_c is the normal fan of P_c by Proposition 2.4, and it is unimodular if and only if $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every maximal face $\sigma \in \Delta_c$. This is equivalent to every $b \in cone(A_\sigma) \cap \mathbb{Z}^d$ lying in $\mathbb{N}A_\sigma$ for every maximal face σ of Δ_c. By Lemma 2.3, this happens if and only if $LP_{A,c}(b)$ has an integral optimum for all $b \in cone(A) \cap \mathbb{Z}^d$. □

For an algebraic algorithm to check TDI-ness see [**A5**] in Section 8.

Theorem 6.8. *If $yA \leq c$ is TDI then $IP_{A,c}$ is a Gomory family.*

Proof. By Theorem 4.3, $(0, \sigma)$ is a standard pair of $\mathcal{O}_c$ for every maximal face σ of Δ_c. Lemma 6.7 implies that $cone(A_\sigma)$ is unimodular (i.e., $\mathbb{Z}A_\sigma = \mathbb{Z}^d$), and therefore $\mathbb{N}A_\sigma = cone(A_\sigma) \cap \mathbb{Z}^d$ for every maximal face σ of Δ_c. Hence the semigroups $\mathbb{N}A_\sigma$ arising from the standard pairs $(0, \sigma)$ as σ varies over the maximal faces of Δ_c cover $\mathbb{N}A$. Therefore the only standard pairs of $\mathcal{O}_c$ are $(0, \sigma)$ as σ varies over the maximal faces of Δ_c. The result then follows from Theorem 6.2. □

When $yA \leq c$ is TDI, the multiplicity of a maximal face of Δ_c in $\mathcal{O}_c$ is one (from Theorem 4.5). By Theorem 6.8, no lower dimensional face of Δ_c is associated to $IP_{A,c}$. While this is sufficient for $IP_{A,c}(b)$ to be a Gomory family, it is far from necessary. TDI-ness guarantees *local integrality* in the sense that $LP_{A,c}(b)$ has an integral optimum for every integral b in $cone(A)$. In contrast, if $IP_{A,c}$ is a Gomory family, the linear optima of the programs in $LP_{A,c}$ may not be integral.

If A is *unimodular* (i.e., $\mathbb{Z}A_\sigma = \mathbb{Z}^d$ for every nonsingular maximal submatrix A_σ of A), then the feasible regions of the linear programs in $LP_{A,c}$ have integral vertices for each $b \in cone(A) \cap \mathbb{Z}^d$, and $yA \leq c$ is TDI *for all c*. Hence if A is unimodular, then $IP_{A,c}$ is a Gomory family for all generic cost vectors c. However, just as integrality of the optimal solutions of programs in $LP_{A,c}$ is not necessary for $IP_{A,c}$ to be a Gomory family, unimodularity of A is not necessary for $IP_{A,c}$ to be a Gomory family *for all c*.

Example 6.9. Consider the seven by twelve integer matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

of rank seven. The maximal minors of A have absolute values zero, one and two and hence A is not unimodular. This matrix has 376 distinct regular triangulations supporting 418 distinct order ideals $\mathcal{O}_c$ (computed using TiGERS). In each case, the standard pairs of $\mathcal{O}_c$ are indexed by just the maximal simplices of the regular triangulation Δ_c that supports it. Hence $IP_{A,c}$ is a Gomory family for all generic c. □

The above discussion shows that $IP_{A,c}$ being a Gomory family is more general than $yA \leq c$ being TDI. Similarly, $IP_{A,c}$ being a Gomory family for all generic c is more general than A being a unimodular matrix.

7. Gomory Families and Hilbert Bases

As we just saw, unimodular matrices or more generally, unimodular regular triangulations lead to Gomory families. A common property of unimodular matrices and matrices A such that $cone(A)$ has a unimodular triangulation is that the columns of A form a *Hilbert basis* for $cone(A)$, i.e., $\mathbb{N}A = cone(A) \cap \mathbb{Z}^d$ (assuming $\mathbb{Z}A = \mathbb{Z}^d$).

Definition 7.1. A $d \times n$ integer matrix A is *normal* if the semigroup $\mathbb{N}A$ equals $cone(A) \cap \mathbb{Z}^d$.

The reason for this (highly over used) terminology here is that if the columns of A form a Hilbert basis, then the zero set of the toric ideal I_A (called a *toric variety*) is a *normal* variety. See [39, Chapter 14] for more details. We first note that if A is not normal, then $IP_{A,c}$ need not be a Gomory family for any cost vector c.

Example 7.2. The matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$ is not normal since $(1,2)^t$ which lies in $cone(A) \cap \mathbb{Z}^2$ cannot be written as a non-negative integer combination of the columns of A. This matrix gives rise to 10 distinct order ideals $\mathcal{O}_c$ supported on its four regular triangulations $\{\{1,4\}\}, \{\{1,2\},\{2,4\}\}, \{\{1,3\},\{3,4\}\}$ and $\{\{1,2\},\{2,3\},\{3,4\}\}$. Each $\mathcal{O}_c$ has at least one standard pair that is indexed by a lower dimensional face of Δ_c.

The matrix in Example 4.2 is also not normal and has no Gomory families. While we do not know whether normality of A is sufficient for the existence of a generic cost vector c such that $IP_{A,c}$ is a Gomory family, we will now show that under certain additional conditions, normal matrices do give rise to Gomory families.

Definition 7.3. A $d \times n$ integer matrix A is Δ-normal if $cone(A)$ has a triangulation Δ such that for every maximal face $\sigma \in \Delta$, the columns of A in $cone(A_\sigma)$ form a Hilbert basis.

Remark 7.4. If A is Δ-normal for some triangulation Δ, then it is normal. To see this note that every lattice point in $cone(A)$ lies in $cone(A_\sigma)$ for some maximal face $\sigma \in \Delta$. Since A is Δ-normal, this lattice point also lies in the semigroup generated by the columns of A in $cone(A_\sigma)$ and hence in $\mathbb{N}A$.

Observe that A is Δ-normal with respect to all the unimodular triangulations of $cone(A)$. Hence triangulations Δ with respect to which A is Δ-normal generalize unimodular triangulations of $cone(A)$.

Problem 7.5. *Are there known families of integer programs whose coefficient matrices are normal or Δ-normal but not unimodular ? Are there known Gomory families of integer programs in the literature (not arising from unimodular matrices) ?*

Examples 7.6 and 7.7 show that the set of matrices where $cone(A)$ has a unimodular triangulation is a proper subset of the set of Δ-normal matrices which in turn is a proper subset of the set of normal matrices.

Example 7.6. Examples of normal matrices with no unimodular triangulations can be found in [7] and [13]. If $cone(A)$ is simplicial for such a matrix, A will be Δ-normal with respect to its coarsest (regular) triangulation Δ consisting of the single maximal face with support $cone(A)$. For instance, consider the following example taken from [13]:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Here $cone(A)$ has 77 regular triangulations and no unimodular triangulations. Since $cone(A)$ is a simplicial cone generated by a_1, a_2, a_3 and a_8, A is Δ-normal with respect to its coarsest regular triangulation $\Delta = \{\{1, 2, 3, 8\}\}$.

Example 7.7. There are normal matrices A that are not Δ-normal with respect to any triangulation of $cone(A)$. To see such an example, consider the following modification of the matrix in Example 7.6 that appears in [39, Example 13.17] :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This matrix is again normal and each of its nine columns generate an extreme ray of $cone(A)$. Hence the only way for this matrix to be Δ-normal for some Δ would be if Δ is a unimodular triangulation of $cone(A)$. However, there are no unimodular triangulations of this matrix.

Theorem 7.8. *If A is Δ-normal for some regular triangulation Δ then there exists a generic cost vector $c \in \mathbb{Z}^n$ such that $\Delta = \Delta_c$ and $IP_{A,c}$ is a Gomory family.*

Proof. Without loss of generality we can assume that the columns of A in $cone(A_\sigma)$ form a minimal Hilbert basis for every maximal face σ of Δ. If there were a redundant element, the smaller matrix obtained by removing this column from A would still be Δ-normal.

For a maximal face $\sigma \in \Delta$, let $\sigma_{in} \subset \{1, \ldots, n\}$ be the set of indices of all columns of A lying in $cone(A_\sigma)$ that are different from the columns of A_σ. Suppose $a_{i_1}, \ldots, a_{i_k}$ are the columns of A that generate the one dimensional faces of Δ, and $c' \in \mathbb{R}^n$ a cost vector such that $\Delta = \Delta_{c'}$. We modify c' to obtain a new cost vector $c \in \mathbb{R}^n$ such that $\Delta = \Delta_c$ as follows. For $j = 1, \ldots, k$, let $c_{i_j} := c'_{i_j}$. If $j \in \sigma_{in}$ for some maximal face $\sigma \in \Delta$, then $a_j = \sum_{i \in \sigma} \lambda_i a_i$, $0 \leq \lambda_i < 1$ and we define $c_j := \sum_{i \in \sigma} \lambda_i c_i$. Hence, for all $j \in \sigma_{in}$, $(a_j^t, c_j) \in \mathbb{R}^{d+1}$ lies in $C_\sigma := cone((a_i^t, c_i) : i \in \sigma) = cone((a_i^t, c'_i) : i \in \sigma)$ which was a facet of $C = cone((a_i^t, c'_i) : i = 1, \ldots, n)$. If $y \in \mathbb{R}^d$ is a vector as in Definition 2.1 showing that σ is a maximal face of $\Delta_{c'}$ then $y \cdot a_i = c_i$ for all $i \in \sigma \cup \sigma_{in}$ and $y \cdot a_j < c_j$ otherwise. Since $cone(A_\sigma) = cone(A_{\sigma \cup \sigma_{in}})$, we conclude that $cone(A_\sigma)$ is a maximal face of Δ_c.

If $b \in \mathbb{N}A$ lies in $cone(A_\sigma)$ for a maximal face $\sigma \in \Delta_c$, then $IP_{A,c}(b)$ has at least one feasible solution u with support in $\sigma \cup \sigma_{in}$ since A is Δ-normal. Further, $(b^t, c \cdot u) = ((Au)^t, c \cdot u)$ lies in C_σ and all feasible solutions of $IP_{A,c}(b)$ with support in $\sigma \cup \sigma_{in}$ have the same cost

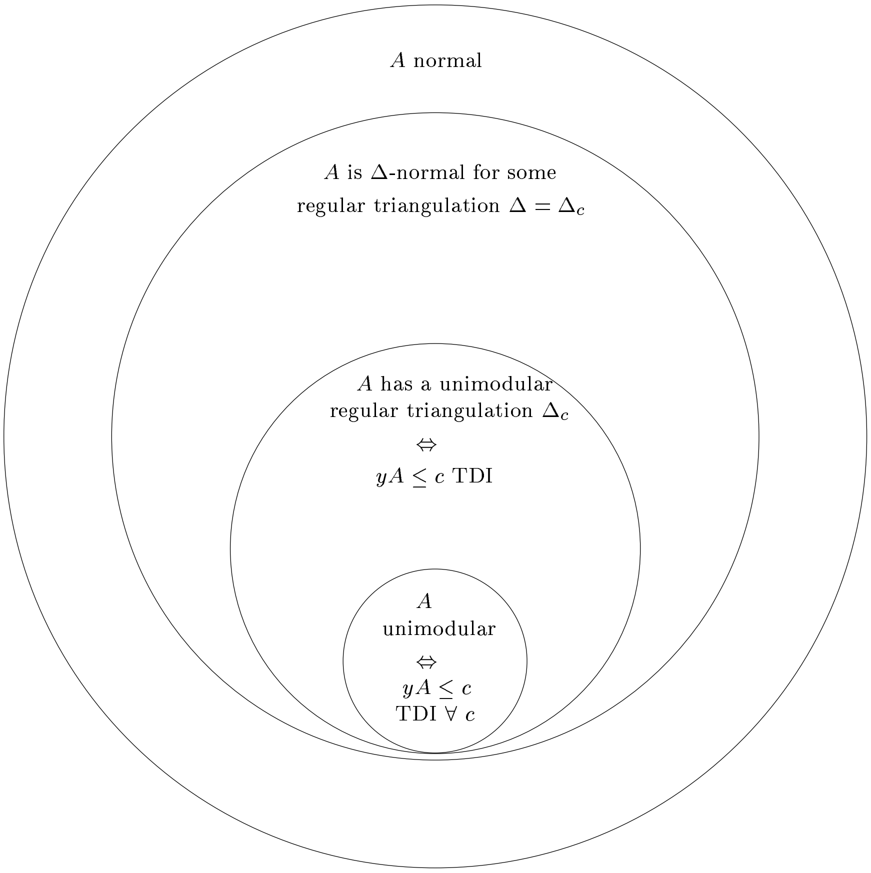

FIGURE 8. Inclusions of sets of matrices

value by construction. Suppose $v \in \mathbb{N}^n$ is any feasible solution of $IP_{A,c}(b)$ with support not in $\sigma \cup \sigma_{in}$. Then $c \cdot u < c \cdot v$ since $(a_i^t, c_i) \in C_\sigma$ if and only if $i \in \sigma \cup \sigma_{in}$ and C_σ is a lower facet of C. Hence the optimal solutions of $IP_{A,c}(b)$ are precisely those feasible solutions with support in $\sigma \cup \sigma_{in}$. The vector b can be expressed as $b = b' + \sum_{i \in \sigma} z_i a_i$ where $z_i \in \mathbb{N}$ are unique and $b' \in \{\sum_{i \in \sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ is also unique. The vector $b' = \sum_{j \in \sigma_{in}} r_j a_j$ where $r_j \in \mathbb{N}$. Setting $u_i = z_i$ for all $i \in \sigma$, $u_j = r_j$ for all $j \in \sigma_{in}$ and $u_k = 0$ otherwise, we obtain all feasible solutions u of $IP_{A,c}(b)$ with support in $\sigma \cup \sigma_{in}$.

If there is more than one such feasible solution, then c is not generic. In this case, we can perturb c to a generic cost vector $c'' = c + \epsilon\omega$ by choosing $1 \gg \epsilon > 0$, $\omega_j \ll 0$ whenever $j = i_1, \ldots, i_k$ and $\omega_j = 0$ otherwise. Suppose $u_1, \ldots, u_t$ are the optimal solutions of the integer programs $IP_{A,c''}(b')$ where $b' \in \{\sum_{i \in \sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$. (Note that $t = |\{\sum_{i \in \sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d|$ is the index of $\mathbb{Z}A_\sigma$ in $\mathbb{Z}A$.) The support of each such u_i is contained in σ_{in}. For any $b \in cone(A_\sigma) \cap \mathbb{Z}^d$, the optimal solution of $IP_{A,c''}(b)$ is hence $u = u_i + z$ for some $i \in \{1, \ldots, t\}$ and $z \in \mathbb{N}^n$ with support in σ. This shows that $\mathbb{N}A$ is covered by the affine semigroups $\phi_A(S(u_i, \sigma))$ where σ is a maximal face of Δ and u_i as above for each σ. By construction, the corresponding admissible pairs (u_i, σ) are all standard for

$\mathcal{O}_{c''}$. Since all data is integral, $c'' \in \mathbb{Q}^n$ and hence can be scaled to lie in $\mathbb{Z}^n$. Renaming c'' as c, we conclude that $IP_{A,c}$ is a Gomory family. □

Corollary 7.9. *Let A be a normal matrix such that $cone(A)$ is simplicial, and let Δ be the coarsest triangulation whose single maximal face has support $cone(A)$. Then there exists a cost vector $c \in \mathbb{Z}^n$ such that $\Delta = \Delta_c$ and $IP_{A,c}$ is a Gomory family.*

Example 7.10. Consider the normal matrix in Example 6.3. Here $cone(A)$ is generated by the first, second and sixth columns of A and hence A is Δ-normal with respect to the regular triangulation $\{\{1, 2, 6\}\}$. There are 13 distinct sets $\mathcal{O}_c$ supported on Δ. Among the 13 corresponding families of integer programs, only one is a Gomory family. A representative cost vector for this $IP_{A,c}$ is $c = (0, 0, 4, 4, 1, 0)$. The standard pair decomposition of $\mathcal{O}_c$ is the one constructed in Theorem 7.8. The affine semigroups $S(\cdot, \sigma)$ from this decomposition are:

$$S(0, \sigma),\ S(e_3, \sigma),\ S(e_4, \sigma),\ \text{and}\ S(e_5, \sigma).$$

Note that A is not Δ-normal with respect to the regular triangulation supporting the Gomory family $IP_{A,c}$ in Example 6.3. The columns of A in $cone(A_{\sigma_1})$ are the columns of A_{σ_1} and A_3. The vector (1, 2, 2) is in the minimal Hilbert basis of $cone(A_{\sigma_1})$ but is not a column of A. This example shows that a regular triangulation Δ of $cone(A)$ can support a Gomory family even if A is not Δ-normal. The Gomory families in Theorem 7.8 have a very special standard pair decomposition. □

Problem 7.11. *If $A \in \mathbb{Z}^{d\times n}$ is a normal matrix, does there exist a generic cost vector $c \in \mathbb{Z}^n$ such that $IP_{A,c}$ is a Gomory family?*

While we do not know the answer to this question, we will now show that stronger results are possible for small values of d.

Theorem 7.12. *If $A \in \mathbb{Z}^{d\times n}$ is a normal matrix and $d \leq 3$, then there exists a generic cost vector $c \in \mathbb{Z}^n$ such that $IP_{A,c}$ is a Gomory family.*

Proof. It is known that if $d \leq 3$ then $cone(A)$ has a regular unimodular triangulation Δ_c [37]. The result then follows from Corollary 6.8. □

Before we proceed, we rephrase Problem 7.11 in terms of covering properties of $cone(A)$ and $\mathbb{N}A$ along the lines of [7], [8], [9], [13] and [37]. To obtain the same set up as in these papers we assume in this section that A is normal and the columns of A form the unique minimal Hilbert basis of $cone(A)$. Using the terminology in [8], the *free Hilbert cover* problem asks whether there exists a covering of $\mathbb{N}A$ by semigroups $\mathbb{N}A_\tau$ where the columns of A_τ are linearly independent. The *unimodular Hilbert cover* problem asks whether $cone(A)$ can be covered by full dimensional unimodular subcones $cone(A_\tau)$ (i.e., $\mathbb{Z}A_\tau = \mathbb{Z}^d$), while the stronger *unimodular Hilbert partition* problem asks whether $cone(A)$ has a unimodular triangulation. (Note that if $cone(A)$ has a unimodular Hilbert cover or partition using subcones $cone(A_\tau)$, then $\mathbb{N}A$ is covered by the semigroups $\mathbb{N}A_\tau$.) All these problems have positive answers if $d \leq 3$ since $cone(A)$ admits a unimodular Hilbert partition in this case [7], [37]. Normal matrices (with $d = 4$) such that $cone(A)$ has no unimodular Hilbert partition can be found in [7] and [13]. Examples (with $d = 6$) that admit no free Hilbert cover and hence no unimodular Hilbert cover can be found in [8] and [9].

When $yA \leq c$ is TDI, the standard pair decomposition of $\mathbb{N}A$ induced by c gives a unimodular Hilbert partition of $cone(A)$ by Theorem 6.7. An important difference between Problem 7.11 and the Hilbert cover problems is that *affine* semigroups cannot be used in Hilbert covers. Moreover, affine semigroups that are allowed in standard pair decompositions come from integer programming. If there are no restrictions on the affine semigroups that can be used in a cover, $\mathbb{N}A$ can always be covered by full dimensional affine semigroups: for any triangulation Δ of $cone(A)$ with maximal subcones $cone(A_\sigma)$, the affine semigroups $b + \mathbb{N}A_\sigma$ cover $\mathbb{N}A$ as b varies in $\{\sum_{i\in\sigma} \lambda_i a_i \,:\, 0 \leq \lambda_i < 1\} \cap \mathbb{Z}^d$ and σ varies among the maximal faces of the triangulation. A *partition* of $\mathbb{N}A$ derived from this idea can be found in [38, Theorem 5.2]. We recall the notion of *supernormality* introduced in [22].

Definition 7.13. A matrix $A \in \mathbb{Z}^{d\times n}$ is *supernormal* if for every submatrix A' of A, the columns of A that lie in $cone(A')$ form a Hilbert basis for $cone(A')$.

Proposition 7.14. *For $A \in \mathbb{Z}^{d\times n}$, the following are equivalent:*

(i) *A is supernormal,*
(ii) *A is Δ-normal for every regular triangulation Δ of $cone(A)$,*
(iii) *Every triangulation of $cone(A)$ in which all columns of A generate one dimensional faces is unimodular.*

Proof. The equivalence of (i) and (iii) was established in [22, Proposition 3.1]. Definition 7.13 shows that (i) $\Rightarrow$ (ii). Hence we just need to show that (ii) $\Rightarrow$ (i). Suppose that A is Δ-normal for every regular triangulation of $cone(A)$. In order to show that A is supernormal we only need to check submatrices A' where the dimension of $cone(A')$ is d. Choose a cost vector c with $c_i \gg 0$ if the ith column of A does not generate an extreme ray of $cone(A')$, and $c_i = 0$ otherwise. This gives a polyhedral subdivision of $cone(A)$ in which $cone(A')$ is a maximal face. There are standard procedures that will refine this subdivision to a regular triangulation Δ of $cone(A)$. Let T be the set of maximal faces σ of Δ such that $cone(A_\sigma)$ lies in $cone(A')$. Since A is Δ-normal, the columns of A that lie in $cone(A_\sigma)$ form a Hilbert basis for $cone(A_\sigma)$ for each $\sigma \in T$. However, since their union is the set of columns of A that lie in $cone(A')$, this union forms a Hilbert basis for $cone(A')$. □

It is easy to catalog all Δ-normal and supernormal matrices, of the type considered in this paper, for small values of d. We say that the matrix A is *graded* if its columns span an affine hyperplane in $\mathbb{R}^d$. If $d = 1$, $cone(A)$ has n triangulations $\{\{i\}\}$ each of which has the unique maximal subcone $cone(a_i)$ whose support is $cone(A)$. If we assume that $a_1 \leq a_2 \leq \cdots \leq a_n$, then A is normal if and only if either $a_1 = 1$, or $a_n = -1$. Also, A is normal if and only if it is supernormal. If $d = 2$ and the columns of A are ordered counterclockwise around the origin, then A is normal if and only if $det(a_i, a_{i+1}) = 1$ for all $i = 1, \ldots, n-1$. Such an A is supernormal since it is Δ-normal for every triangulation Δ — the Hilbert basis of a maximal subcone of Δ is precisely the set of columns of A in that subcone. If $d = 3$ then as mentioned before, $cone(A)$ has a unimodular triangulation with respect to which A is Δ-normal. However, not every such A needs to be supernormal: the matrix in Example 6.3 is not Δ-normal for the Δ supporting the Gomory family in that example. If $d = 3$ and A

is graded, then without loss of generality we can assume that the columns of A span the hyperplane $x_1 = 1$. If A is normal as well, then its columns are precisely all the lattice points in the convex hull of A. Conversely, every graded normal A with $d = 3$ arises this way — its columns are all the lattice points in a polygon in $\mathbb{R}^2$ with integer vertices. In particular, every triangulation of $cone(A)$ that uses all the columns of A is unimodular. Hence, by Proposition 7.14, A is supernormal, and therefore Δ-normal for any triangulation of A.

Theorem 7.15. *Let $A \in \mathbb{Z}^{d\times n}$ be a normal matrix of rank d.*

(i) *If $d = 1, 2$ or A is graded and $d = 3$, every regular triangulation of $cone(A)$ supports at least one Gomory family.*
(ii) *If $d = 2$ and A is graded, every regular triangulation of $cone(A)$ supports exactly one Gomory family.*
(iii) *If $d = 3$ and A is not graded, or if $d = 4$ and A is graded, then not all regular triangulations of $cone(A)$ may support a Gomory family. In particular, A may not be Δ-normal with respect to every regular triangulation.*

Proof. (i) If $d = 1, 2$ or A is graded and $d = 3$, A is supernormal and hence by Proposition 7.14 and Theorem 7.8, every regular triangulation of $cone(A)$ supports at least one Gomory family.
(ii) If $d = 2$ and A is graded, then we may assume that

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n-1 \end{bmatrix}.$$

In this case, A is supernormal and hence every regular triangulation Δ of $cone(A)$ supports a Gomory family by Theorem 7.8. Suppose the maximal cones of Δ, in counter-clockwise order, are $C_1, \dots, C_r$. Assume the columns of A are labeled such that $C_i = cone(a_{i-1}, a_i)$ for $i = 1, \dots, r$, and the columns of A in the interior of C_i are labeled in counter-clockwise order as $b_{i1}, \dots, b_{ik_i}$. Hence the n columns of A from left to right are:

$$a_0, b_{11}, \cdots, b_{1k_1}, a_1, b_{21}, \cdots, a_{r-1}, b_{r1}, \cdots, b_{rk_r}, a_r.$$

Indexing the columns of A by their labels, the maximal faces of Δ are $\sigma_i = \{i-1, i\}$ for $i = 1, \dots, r$. Let e_i be the unit vector of $\mathbb{R}^n$ indexed by the true column index of a_i in A and e_{ij} be the unit vector of $\mathbb{R}^n$ indexed by the true column index of b_{ij} in A. Since the columns of A form a minimal Hilbert basis of $cone(A)$, e_i is the unique solution to $IP_{A,c}(a_i)$ for all c and e_{ij} is the unique solution to $IP_{A,c}(b_{ij})$ for all c. Hence the standard pairs of Theorem 7.8 are $(0, \sigma_i)$ and (e_{ij}, σ_i) for $i = 1, \dots, r$ and $j = 1, \dots, k_i$.

Suppose Δ supports a second Gomory family $IP_{A,\omega}$. Then every standard pair of $\mathcal{O}_w$ is also of the form $(*, \sigma_i)$ for $\sigma_i \in \Delta$, and r of them are $(0, \sigma_i)$ for $i = 1, \dots, r$. The remaining standard pairs are of the form (e_{ij}, σ_k). To see this, consider the semigroups in $\mathbb{N}A$ arising from the standard pairs of $\mathcal{O}_w$. The total number of standard pairs of $\mathcal{O}_c$ and $\mathcal{O}_w$ are the same. Since the columns of A all lie on $x_1 = 1$, no two b_{ij}s can be covered by a semigroup coming from the same standard pair and none of them are covered by a semigroup $(0, \sigma_i)$. We show that if (e_{ij}, σ_k) is a standard pair of $\mathcal{O}_w$ then $k = i$ and thus $\mathcal{O}_w = \mathcal{O}_c$.

If $r = 1$, the standard pairs of $\mathcal{O}_w$ are $(0, \sigma_1), (e_{11}, \sigma_1), \dots, (e_{1k_1}, \sigma_1)$ as in Theorem 7.8. If $r > 1$, consider the last cone $C_r = cone(a_{r-1}, a_r)$. If a_{r-1} is the second to last column of A, then C_r is unimodular and the semigroup from $(0, \sigma_r)$ covers $C_r \cap \mathbb{Z}^2$. The subcomplex comprised of $C_1, \ldots, C_{r-1}$ is a regular triangulation Δ' of $cone(A')$ where A' is obtained by dropping the last column of A. Since A' is a normal graded matrix with $d = 2$ and Δ' has less than r maximal cones, the standard pairs supported on Δ' are as in Theorem 7.8 by induction. If a_{r-1} is not the second to last column of A then b_{rk_r}, the second to last column of A is in the Hilbert basis of C_r but is not a generator of C_r. So $\mathcal{O}_w$ has a standard pair of the form (e_{rk_r}, σ_i). If $\sigma_i \neq \sigma_r$, then the lattice point $b_{rk_r} + a_r$ cannot be covered by the semigroup from this or any other standard pair of $\mathcal{O}_w$. Hence $\sigma_i = \sigma_r$. By a similar argument, the remaining standard pairs indexed by σ_r are $(e_{r(k_r-1)}, \sigma_r), \ldots, (e_{r1}, \sigma_r)$ along with $(0, \sigma_r)$. These are precisely the standard pairs of $\mathcal{O}_c$ indexed by σ_r. Again we are reduced to considering the subcomplex comprised of $C_1, \ldots, C_{r-1}$ and by induction, the remaining standard pairs of $\mathcal{O}_w$ are as in Theorem 7.8.

(iii) The 3×6 normal matrix A of Example 6.3 has 10 distinct Gomory families supported on 10 out of the 14 regular triangulations of $cone(A)$. Furthermore, the normal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

has 11 distinct Gomory families supported on 11 out of its 19 regular triangulations. □

8. Algebraic Notes

[A1]: A *monomial* x^u in the *polynomial ring* $S := k[x_1, \ldots, x_n]$ is a product $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ where $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$. We assume that k is a field, say the set of rational numbers. For a scalar $k_u \in k$ and a monomial x^u in S, we call $k_u x^u$ a *term* of S. A *polynomial* $f = \sum k_u x^u$ in S is a combination of finitely many terms in S. A subset I of S is an *ideal* of S if (1) I is closed under addition, i.e., $f, g \in I \Rightarrow f + g \in I$ and (2) if $f \in I$ and $g \in S$ then $fg \in I$. We say that I is generated by the polynomials $f_1, \ldots, f_t$, denoted as $I = \langle f_1, \ldots, f_t \rangle$, if $I = \{\sum_{i=1}^t f_i g_i \ : \ g_i \in S\}$. By Hilbert's basis theorem, every ideal in S has a finite generating set. An ideal M in S is called a *monomial ideal* if it is generated by monomials, i.e., $M = \langle x^{v_1}, \ldots, x^{v_t} \rangle$ for monomials $x^{v_1}, \ldots, x^{v_t}$ in S. The monomials that do not lie in M are called the *standard monomials* of M. The *cost* of a term $k_u x^u$ with respect to a vector $c \in \mathbb{R}^n$ is the dot product $c \cdot u$. The *initial term* of a polynomial $f = \sum k_u x^u \in S$ with respect to c, denoted as $in_c(f)$, is the sum of all terms in f of maximal cost. For any ideal $I \subset S$, the *initial ideal* of I with respect to c, denoted as $in_c(I)$, is the ideal generated by all the initial terms $in_c(f)$ of all polynomials f in I. These concepts come from the theory of *Gröbner bases* for polynomial ideals. See [10] for an introduction.

The *toric ideal* of the matrix A, denoted as I_A, is the *binomial* ideal in S defined as:

$$I_A := \langle x^u - x^v : \ u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle.$$

Toric ideals provide the link between integer programming and Gröbner basis theory. See [39] and [44] for an introduction to this area of research. This connection yields the following basic facts that we state without proofs.

Lemma 8.1. [39] *(i) If c is generic, then the initial ideal $in_c(I_A)$ is a monomial ideal.*
(ii) A lattice point u is non-optimal for the integer program $IP_{A,c}(Au)$, or equivalently, $u \in \mathcal{N}_c$, if and only if x^u lies in the initial ideal $in_c(I_A)$. In other words, a lattice point u lies in $\mathcal{O}_c$ if and only if x^u is a standard monomial of $in_c(I_A)$.
(iii) The **reduced Gröbner basis** *$\mathcal{G}_c$ of I_A with respect to c is the unique minimal test set for the family of integer programs $IP_{A,c}$.*
(iv) If u is a feasible solution of $IP_{A,c}(b)$, and x^{u^*} is the unique **normal form** *of x^u with respect to $\mathcal{G}_c$, then u^* is the optimal solution of $IP_{A,c}(b)$.*

The above lemma provides an algorithm to compute $\mathcal{O}_c$ algebraically as the lattice points in it are the exponents of the standard monomials of $in_c(I_A)$. This initial ideal is a byproduct of the computation of the reduced Gröbner basis of I_A with respect to the cost vector c. Gröbner bases of polynomial ideals can be computed by Buchberger's algorithm [10].
Example 3.2 continued. In this example, the toric ideal $I_A = \langle x_1^4 - x_3, x_2^2 - x_1x_3\rangle$ and its initial ideal with respect to the cost vector $c = (10000, 100, 1)$ is

$$in_c(I_A) = \langle x_2^8,\ x_1x_3,\ x_1x_2^6,\ x_1^2x_2^4,\ x_1^3x_2^2,\ x_1^4\rangle.$$

Note that the exponent vectors of the generators of $in_c(I_A)$ are the generators of $\mathcal{N}_c$. $\square$

[A2]: A *primary* ideal J in $k[x_1, \ldots, x_n]$ is a proper ideal such that $fg \in J$ implies either $f \in J$ or $g^t \in J$ for some positive integer t. A *prime* ideal J of $k[x_1, \ldots, x_n]$ is a proper ideal such that $fg \in J$ implies that either $f \in J$ or $g \in J$. A *primary decomposition* of an ideal I in $k[x_1, \ldots, x_n]$ is an expression of I as a finite intersection of *primary ideals* in $k[x_1, \ldots, x_n]$. Lemma 3.3 in [41] shows that every monomial ideal M in $k[x_1, \ldots, x_n]$ admits a primary decomposition into irreducible primary ideals that are indexed by the standard pairs of M. The *radical* of an ideal $I \subset k[x_1, \ldots, x_n]$ is the ideal $\sqrt{I} := \{f \in S \ :\ f^t \in I, \text{ for some positive integer } t\}$. Radicals of primary ideals are prime. The radicals of the primary ideals in a minimal primary decomposition of an ideal I are called the *associated primes* of I. This list of prime ideals is independent of the primary decomposition of the ideal. The minimal elements among the associated primes of I are called the *minimal primes* of I while the others are called the *embedded primes* of I. The minimal primes of I are precisely the defining ideals of the isolated components of the zero-set or *variety* of I while the embedded primes cut out embedded subvarieties in the isolated components. See a textbook in commutative algebra like [11] for more details.

A face τ of Δ_c is an associated set of $IP_{A,c}$ if and only if the monomial prime ideal $p_\tau := \langle x_j \ :\ j \notin \tau\rangle$ is an associated prime of the ideal $in_c(I_A)$. Further, p_σ is a minimal prime of $in_c(I_A)$ if and only if σ is a maximal face of Δ_c. Hence the lower dimensional associated sets of $IP_{A,c}$ index the embedded primes of $in_c(I_A)$. The standard pair decomposition of a monomial ideal was introduced in [41] to study its associated primes. The multiplicity of an associated prime p_τ of $in_c(I_A)$ is an algebraic invariant of $in_c(I_A)$, and [41] shows that this is exactly the number of standard pairs indexed by τ. Similarly, the arithmetic degree of $in_c(I_A)$ is a refinement of the geometric notion of *degree* and [41] shows that this number is the total number of standard pairs of $in_c(I_A)$. These connections explain our choice of terminology. Theorem 4.3 is a translation of the specialization of Lemma 3.5 in [41] to toric initial ideals. We refer the interested reader to [39, §8 and §12.D] and [41, §3] for the algebraic connections. Theorem 4.5 is a staple result of toric geometry and also follows from [15, Theorem 1]. It is proved via the algebraic technique of *localization* in [39, Theorem 8.8]. Bounds on the arithmetic degree of a general monomial ideal in terms of its dimension and minimal generators can be found in [41, Theorem 3.1]. One hopes that stronger bounds are possible for toric initial ideals.

[A3]: In algebraic language, the chain theorem says that the associated primes of $in_c(I_A)$ occur in saturated chains. This was proved in [25, Theorem 3.1]. When the cost vector c is not generic, $in_c(I_A)$ is no longer a monomial ideal, and its associated primes need not come in saturated chains. See [25, Remark 3.3] for such an example. Algebraically, Problem 3.3 asks for a characterization of all monomial ideals that can appear as a monomial initial ideal of a toric ideal. Theorem 5.2 imposes the necessary condition that the associated primes of such a monomial ideal has to come in saturated chains. Unfortunately, this is not sufficient. See [33] for another class of monomial ideals that also have the chain property.

[A4]: Algebraically, $IP_{A,c}$ is a Gomory family if and only if the initial ideal $in_c(I_A)$ has no embedded primes and hence Theorem 6.2 is a characterization of toric initial ideals without embedded primes. A sufficient condition for an ideal in $k[x_1, \ldots, x_n]$ to not have embedded primes is that it is *Cohen-Macaulay* [11]. In general, Cohen-Macaulayness is not necessary for an ideal to be free of embedded primes. However, empirical evidence seemed to suggest for a while that for toric initial ideals, Cohen-Macaulayness might be equivalent to being free of embedded primes. A counterexample to this was found recently by Laura Matusevich.

[A5]: Corollary 8.4 in [39] shows that Δ_c is unimodular if and only if the monomial ideal $in_c(I_A)$ is generated by *square-free* monomials. Hence, by computing $in_c(I_A)$, one can determine whether $yA \leq c$ is TDI. Such computations can be carried out on computer algebra systems like CoCoA [1] or MACAULAY 2 [21] for moderately sized examples. See [39] for algorithms. Standard pair decompositions of monomial ideals can be computed with MACAULAY 2 [23].

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Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195
E-mail address: thomas@math.washington.edu