Matrix Cubes Parametrized by Eigenvalues

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Focused Research Group
Semidefinite Optimization and Convex Algebraic Geometry
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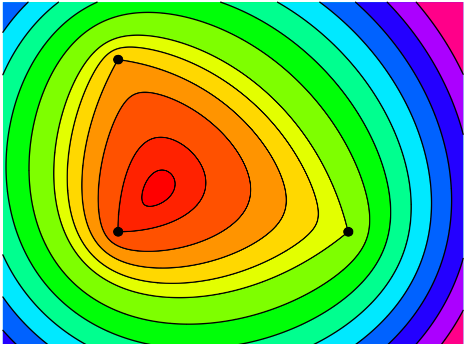
Abstract

"An elimination problem in semidefinite programming is solved by means of tensor algebra. It concerns families of matrix cube problems whose constraints are the minimum and maximum eigenvalue function on an affine space of symmetric matrices. An LMI representation is given for the convex set of all feasible instances, and its boundary is studied from the perspective of algebraic geometry. This generalizes our earlier work with Pablo Parrilo on m-ellipses and m-ellipsoids."



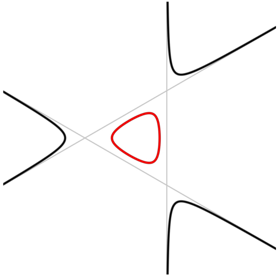
Semidefinite Programming

A *spectrahedron* is the intersection of the cone of positive semidefinite matrices with an affine-linear space. The algebraic representation of such a convex set is a *linear matrix inequality*.

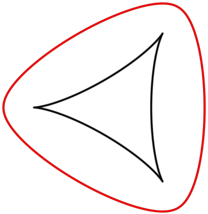
Semidefinite programming is the computational problem of maximizing a linear function over a spectrahedron.

Convex Algebraic Geometry

is the marriage of real algebraic geometry with optimization theory.
It concerns figures such as the Zariski closure of a spectrahedron:



Duality is important in both optimization and algebraic geometry:



Matrix Cubes

Consider the convex semi-algebraic set

$$\mathcal{C} \;=\; \left\{ (x,d) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \begin{array}{l} d \cdot A_0 + \sum_{k=1}^{m} t_k A_k \succeq 0 \;\;\text{whenever} \\ \lambda_{\min}(B_k(x)) \le t_k \le \lambda_{\max}(B_k(x)) \\ \hfill \text{for}\;\; k = 1,2,\ldots,m \end{array} \right\},$$

where the A_i are constant symmetric matrices of size $N_0 \times N_0$, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote minimum and maximum eigenvalues. The m matrices $B_k(x)$ are linear matrix polynomials

$$B_k(x) \;=\; B_0^{(k)} + x_1 B_1^{(k)} + \cdots + x_n B_n^{(k)},$$

where $B_i^{(k)}$ is a constant symmetric $N_k \times N_k$ matrix.

Theorem

The set $\mathcal{C}$ is a spectrahedron. Its LMI has size $N_0 N_1 \cdots N_m$.

Matrix Cube to m-Ellipse

We model the m-ellipse using matrix cubes as follows.
Let $N_0 = 1$ and $N_1 = \cdots = N_m = 2$. Each scalar A_i is 1 and

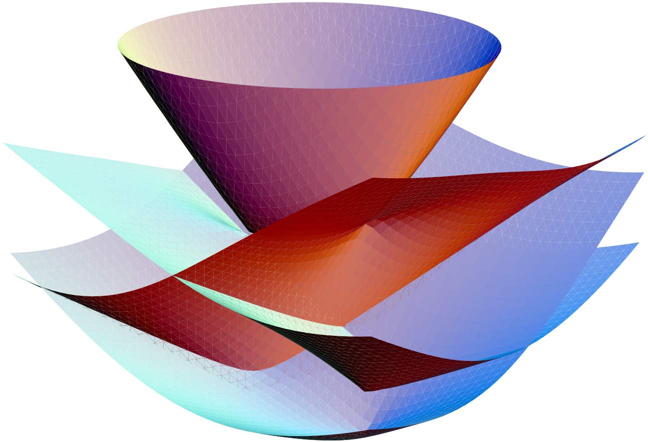
$$B_k(x, y) \;=\; \begin{bmatrix} x - u_k & y - v_k \\ y - v_k & u_k - x \end{bmatrix} \qquad \text{for } \; k = 1, \cdots, m.$$

The eigenvalues of $B_k(x)$ are $\pm\sqrt{(x - u_k)^2 + (y - v_k)^2}$, and

$$\mathcal{C} \;=\; \Big\{ (x, y, d) \in \mathbb{R}^3 \;:\; d \geq t_1 + \cdots + t_m \;\text{ whenever } \; |t_k| \leq \sqrt{(x-u_k)^2 + (y-v_k)^2} \Big\}.$$

This formula characterizes the m-ellipse. The spectrahedron $\mathcal{C}$ consists of all points (x, y, d) such that the sum of the distances from (x, y) to the m points (u_k, v_k) is at most d.

Ellipses with Fixed Foci



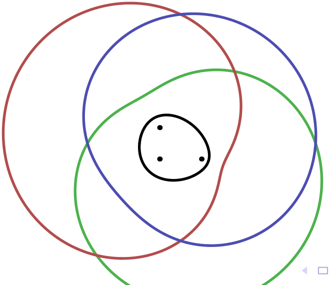
$$\mathcal{C} \;=\; \left\{ (x,y,d) \in \mathbb{R}^3 \,:\, \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \;\leq\; d \right\}.$$

Ellipses are Spectrahedra

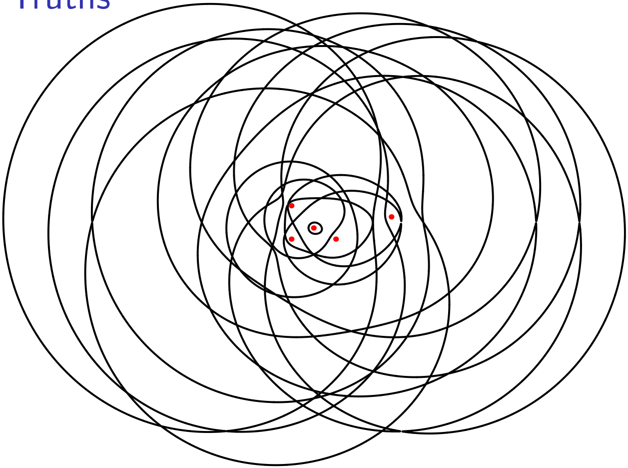
The ellipse with foci $(0,0), (1,0), (0,1)$ has the LMI representation

$$\begin{bmatrix} d+3x-1 & y-1 & y & 0 & y & 0 & 0 & 0 \\ y-1 & d+x-1 & 0 & y & 0 & y & 0 & 0 \\ y & 0 & d+x+1 & y-1 & 0 & 0 & y & 0 \\ 0 & y & y-1 & d-x+1 & 0 & 0 & 0 & y \\ y & 0 & 0 & 0 & d+x-1 & y-1 & y & 0 \\ 0 & y & 0 & 0 & y-1 & d-x-1 & 0 & y \\ 0 & 0 & y & 0 & y & 0 & d-x+1 & y-1 \\ 0 & 0 & 0 & y & 0 & y & y-1 & d-3x+1 \end{bmatrix}$$

It consists of all points (x, y) such that this matrix is positive semidefinite. The boundary is a convex algebraic curve of degree eight:



Algebraic Truths



"The polynomial equation defining the m-ellipse has degree 2^m if m is odd and degree $2^m - \binom{m}{m/2}$ if m is even. We express this polynomial equation as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted m-ellipses and m-ellipsoids in arbitrary dimensions ....."

Research Questions

- What is the genus of the m-ellipse? Singularities?
- What is the algebraic degree of the Fermat-Weber point ?
- Given an LMI representation of a polynomial whose degree is smaller than the matrix size, how to reduce the matrix size?
- Does every compact convex basic semialgebraic set have a lifted LMI representation?
- How to compute the convex hull of a real algebraic variety?
- How to represent the Minkowski sum of two spectrahedra?
- Is there a nice coordinate-free generalization of the *Chordal Matrix Completion Theorem* for spectrahedra?
- Is there a simplex algorithm for SDP?
- How to determine the active constraints and the tangent cone at the optimal solution of an SDP?
- How to tropicalize semidefinite programming ?