1. Let $S$ be the set of all points $(x, y) \in \mathbb{R}$ such that the matrix
\[
\begin{pmatrix}
1 & x & y \\
x & 1 & x \\
y & x & 1
\end{pmatrix}
\]
is positive semidefinite. Draw a picture of the two-dimensional spectrahedron $S$ and its dual convex figure. Find the maximum of the linear function $x + y$ over $S$.

2. Let $S'$ be the set of all points $(x, y) \in \mathbb{R}$ such that the matrix
\[
\begin{pmatrix}
1 & x & x + y \\
x & 1 & y \\
x + y & y & 1
\end{pmatrix}
\]
is positive semidefinite. Draw a picture of the two-dimensional spectrahedron $S'$ and its dual convex figure. Find the maximum of the linear function $x + y$ over $S'$.

3. Fix the four points $(0, 0), (0, 1), (1, 0), (1, 2)$ in the plane $\mathbb{R}^2$. Compute their Fermat-Weber point. Also, compute the irreducible polynomial that vanishes on the 4-ellipse with these foci and arbitrary radius $d > 0$.

4. Let $f(x, y)$ be the polynomial of degree ten computed in the previous exercise, where the radius is now fixed at $d = 17$. Determine all singular points of the complex algebraic curve $\{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \}$. 


5. Fix the five points \((0, 0), (0, 1), (1, 0), (1, 2), (2, 1)\) in the plane \(\mathbb{R}^2\). Compute their Fermat-Weber point \((x^*, y^*)\). Show that \(x^*\) and \(y^*\) are algebraic numbers over \(\mathbb{Q}\), and determine their minimal polynomials.

6. Let \(T\) be a spectrahadron obtained by intersecting the cone of positive semidefinite \(6 \times 6\)-matrices with a generic affine-linear space of dimension 6. What are the possible ranks of extreme points on \(T\)?

7. Consider the unit circle \(C\) and the unit square \(S\) in the plane \(\mathbb{R}^2\). Draw their Minkowski sum \(C + S\). Prove that \(C + S\) is not spectrahedron. Can you express \(C + S\) as the projection of a 3-dimensional spectrahedron?

8. How many straight lines (over \(\mathbb{C}\)) lie on Cayley’s cubic surface?

9. Let \(U\) be any spectrahedron obtained by intersecting the cone of positive semidefinite \(4 \times 4\)-matrices with an affine-linear space of dimension 3. Can you draw \(U\)? How many matrices of rank 2 can \(U\) contain?

10. Maximize the determinant over the spectrahedra \(S\) and \(S'\) in Ex. 1 and 2.

11. Fix the Gaussian graphical model defined by the 4-cycle. Give an example of a sample covariance matrix for which the MLE does not exist. What is the maximal rank of a psd \(4 \times 4\)-matrix with this property?

12. Let \(L\) be the row space of the matrix

\[
A = \begin{pmatrix}
4 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 & 4
\end{pmatrix}
\]

The variety \(L^{-1}\) is a curve in \(\mathbb{P}^4\). Find its prime ideal \(P_L\) and its degree. Explain the geometry and statistical meaning of MLE for this model.

13. Let \(L\) be the space of Hankel matrices of format \(m \times m\) and let \(C_L\) be the spectrahedral cone consisting of all positive definite Hankel matrices. Determine the dual cone \(K_L\) and its boundary hypersurface \(H_L\). What is the projective variety \(L^{-1}\)? What does MLE mean for this model?

14. The **trigonometric moment curve** in \(\mathbb{R}^4\) is given by the parametrization

\[
\theta \mapsto (\cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta)).
\]

Show that this curve is algebraic and compute a Gröbner basis for its prime ideal. Prove that the convex hull of this curve is a spectrahedron. Classify all faces (of all dimensions 1, 2 and 3) of this spectrahedron.