Convex Algebraic Geometry
at the University of Tokyo

Problems for the Short Course by **Bernd Sturmfels**
Wednesday-Thursday, July 8-9, 2009, 13:00-17:00
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1. Let $\mathcal{S}$ be the set of all points $(x, y) \in \mathbb{R}$ such that the matrix

$$\begin{pmatrix} 1 & x & y \\ x & 1 & x \\ y & x & 1 \end{pmatrix} \quad \text{is positive semidefinite.}$$

   Draw a picture of the two-dimensional spectrahedron $\mathcal{S}$ and its dual convex figure. Find the maximum of the linear function $x + y$ over $\mathcal{S}$.

2. Let $\mathcal{S}'$ be the set of all points $(x, y) \in \mathbb{R}$ such that the matrix

$$\begin{pmatrix} 1 & x & x+y \\ x & 1 & y \\ x+y & y & 1 \end{pmatrix} \quad \text{is positive semidefinite.}$$

   Draw a picture of the two-dimensional spectrahedron $\mathcal{S}'$ and its dual convex figure. Find the maximum of the linear function $x + y$ over $\mathcal{S}'$.

3. Fix the four points $(0, 0), (0, 1), (1, 0), (1, 2)$ in the plane $\mathbb{R}^2$. Compute their Fermat-Weber point. Also, compute the irreducible polynomial that vanishes on the 4-ellipse with these foci and arbitrary radius $d > 0$.

4. Let $f(x, y)$ be the polynomial of degree ten computed in the previous exercise, where the radius is now fixed at $d = 17$. Determine all singular points of the complex algebraic curve $\{\, (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \,\}$.

5. Fix the five points $(0,0),(0,1),(1,0),(1,2),(2,1)$ in the plane $\mathbb{R}^2$. Compute their Fermat-Weber point (x^*,y^*). Show that x^* and y^* are algebraic numbers over $\mathbb{Q}$, and determine their minimal polynomials.

6. Let $\mathcal{T}$ be a spectrahadron obtained by intersecting the cone of positive semidefinite 6×6-matrices with a generic affine-linear space of dimension 6. What are the possible ranks of extreme points on $\mathcal{T}$?

7. Consider the unit circle C and the unit square S in the plane $\mathbb{R}^2$. Draw their Minkowski sum $C+S$. Prove that $C+S$ is not spectrahedron. Can you express $C+S$ as the projection of a 3-dimensional spectrahedron?

8. How many straight lines (over $\mathbb{C}$) lie on Cayley's cubic surface ?

9. Let $\mathcal{U}$ be any spectrahedron obtained by intersecting the cone of positive semidefinite 4×4-matrices with an affine-linear space of dimension 3. Can you draw $\mathcal{U}$? How many matrices of rank 2 can $\mathcal{U}$ contain?

10. Maximize the determinant over the spectrahedra $\mathcal{S}$ and $\mathcal{S}'$ in Ex. 1 and 2.

11. Fix the Gaussian graphical model defined by the 4-cycle. Give an example of a sample covariance matrix for which the MLE does not exist. What is the maximal rank of a psd 4×4-matrix with this property?

12. Let $\mathcal{L}$ be the row space of the matrix
$$A \;=\; \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$
The variety $\mathcal{L}^{-1}$ is a curve in $\mathbb{P}^4$. Find its prime ideal $P_{\mathcal{L}}$ and its degree. Explain the geometry and statistical meaning of MLE for this model.

13. Let $\mathcal{L}$ be the space of Hankel matrices of format $m\times m$ and let $\mathcal{C}_{\mathcal{L}}$ be the spectrahedral cone consisting of all positive definite Hankel matrices. Determine the dual cone $\mathcal{K}_{\mathcal{L}}$ and its boundary hypersurface $H_{\mathcal{L}}$. What is the projective variety $\mathcal{L}^{-1}$? What does MLE mean for this model?

14. The *trigonometric moment curve* in $\mathbb{R}^4$ is given by the parametrization
$$\theta \;\mapsto\; \big(\cos(\theta),\sin(\theta),\cos(2\theta),\sin(2\theta)\big).$$
Show that this curve is algebraic and compute a Gröbner basis for its prime ideal. Prove that the convex hull of this curve is a spectrahedron. Classify all faces (of all dimensions 1, 2 and 3) of this spectrahedron.