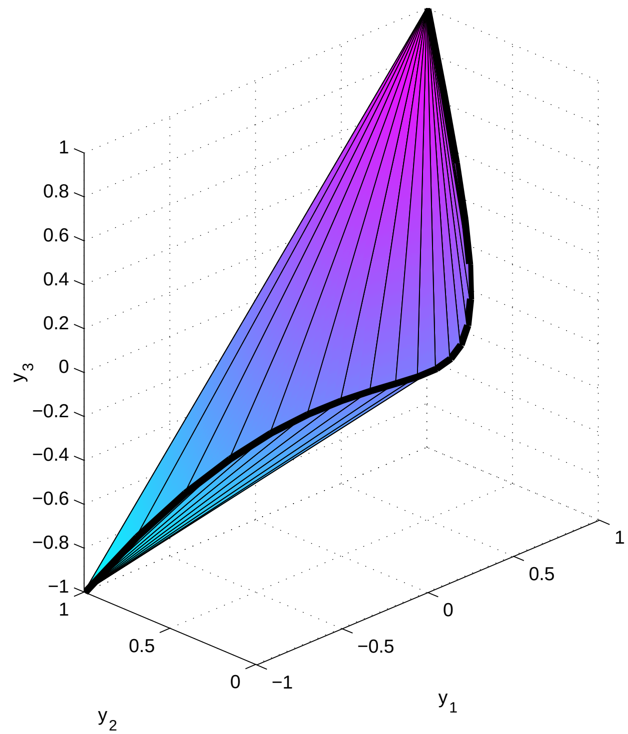
Convex Algebraic Geometry



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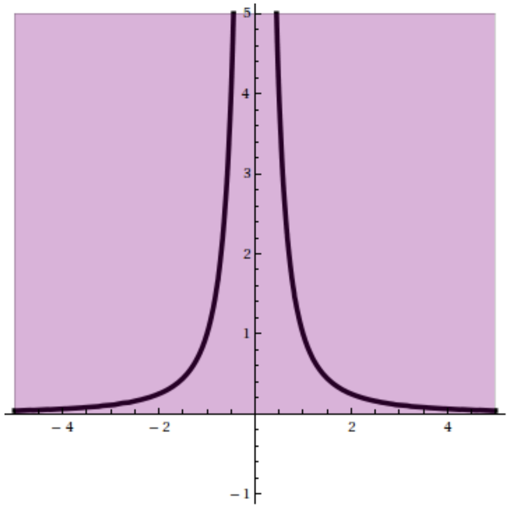
CAG = study of convex hulls of real algebraic varieties

- $I \subseteq \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$ ideal
- $\mathcal{V}_{\mathbb{R}}(I) := \{\mathbf{p} \in \mathbb{R}^n : f(\mathbf{p}) = 0 \ \forall f \in I\}$

  real variety of I, closed, semi-algebraic
- conv$(\mathcal{V}_{\mathbb{R}}(I))$ convex hull of $\mathcal{V}_{\mathbb{R}}(I)$

  convex, semi-algebraic



$I = \langle x^2 y - 1 \rangle$

conv$(\mathcal{V}_{\mathbb{R}}(I))$ open

(1) $S \subseteq \mathbb{R}^n$ is (basic) semi-algebraic if

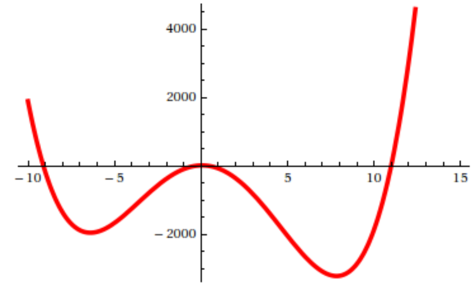
$S = \{\mathbf{p} \in \mathbb{R}^n : f_1(\mathbf{p}) \rhd_1 0, \ldots, f_s(\mathbf{p}) \rhd_s 0\}$,

$f_i \in \mathbb{R}[\mathbf{x}], \quad \rhd_i \in \{\geq, \leq, =, \neq\}$

(2) semi-algebraic set = union of basic semi-algebraic sets

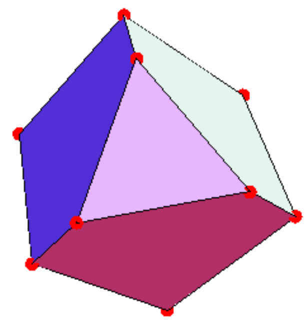
I. Why should anyone care?

(1) univariate ideals: $I = \langle f \rangle$



$\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(f))$ – min and max real roots of f

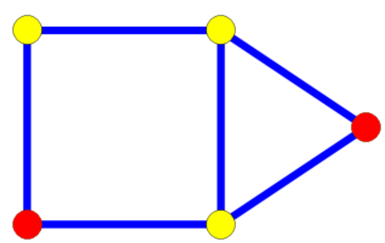
(2) $\mathcal{V}_{\mathbb{R}}(I)$ finite:

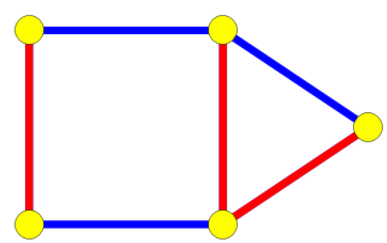


- $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is a polytope
- allows linear optimization over $\mathcal{V}_{\mathbb{R}}(I)$

contains $0/1$ integer programming: max$\{\mathbf{c}\cdot\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \{0,1\}^n\}$

Ex: $G = ([n], E)$ graph on n nodes





(i) max stable set problem in G

(ii) max cut problem in G

(3) Polynomial Optimization: $p^* := \inf\{p(\mathbf{x}) : \mathbf{x} \in K\}$, K semi-algebraic

$$p(\mathbf{x}) = \sum_{\alpha \in S} p_\alpha \mathbf{x}^\alpha, \ |S| = s, \ p_\alpha \in \mathbb{R}$$

- $K = \mathbb{R}^n$ (unconstrained poly optimization):

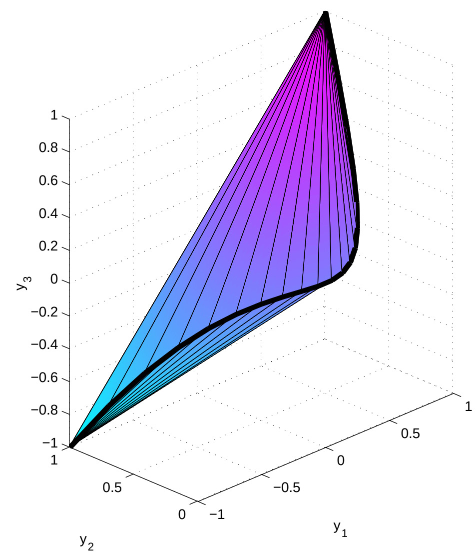
  $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^s, \ \mathbf{t} \mapsto (\mathbf{t}^{\alpha_1}, \ldots, \mathbf{t}^{\alpha_s})$ toric variety

  $p^* = \min\{\sum p_\alpha y_\alpha : y \in \text{conv}(\phi(\mathbb{R}^n))\}$ linear objective!

  $n = 1$ & $\deg(p) = d$: $\Rightarrow$

  $\phi(t) = (1, t, t^2, \ldots, t^d)$

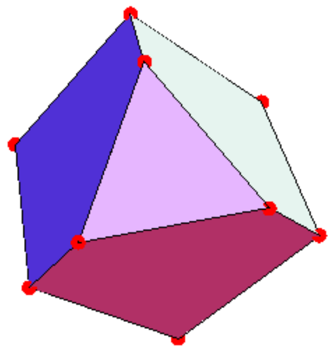
  rational normal curve



- Constrained Optimization:

similar, extensive applications

II. Representing conv$(\mathcal{V}_{\mathbb{R}}(I))$



$\star$ finite real varieties:

conv$(\mathcal{V}_{\mathbb{R}}(I)) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ polytope

- linear programming (polynomial time)
- duality theory / seperation theorem (Farkas lemma)
- projection algorithms (Fourier-Motzkin elimination)
- well developed computational methods

What is a useful representation in other cases?

Driven by available algorithms & their inputs

Semidefinite programming

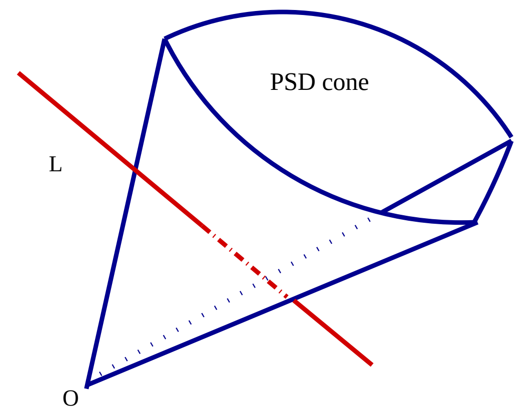
- $A, B \in \text{Symm}(\mathbb{R}^{n\times n})$, $\quad A \cdot B := \text{trace}(AB)$
- $A \succeq 0$ (positive semi-definite)

  $\Leftrightarrow$ all eigenvalues of A are non-negative

  $\Leftrightarrow$ $\mathbf{v}^t A \mathbf{v} \geq 0 \ \ \forall\, \mathbf{v} \in \mathbb{R}^n$

  $\Leftrightarrow$ all principal subdeterminants ≥ 0

  $\Leftrightarrow$ $A = BB^t$ for some $B \in \mathbb{R}^{n\times m}$



semidefinite program (sdp):

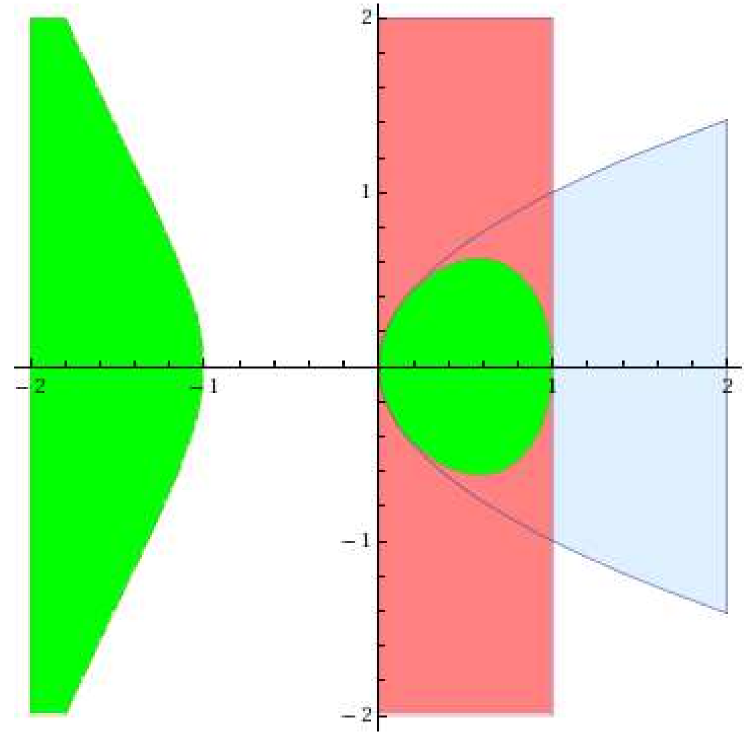
$$\text{sup}\{C \cdot X \ : A_j \cdot X = b_j,\ j = 1, \ldots, m,\ X \succeq 0\}$$

- convex optimization, polynomial time algorithms
- Linear programming is SDP over diagonal matrices
- feasible region – spectrahedron – convex, semi-algebraic

Example:

$$\left\{ (x,y) \in \mathbb{R}^2 : \begin{bmatrix} x & 0 & y \\ 0 & 1 & -x \\ y & -x & 1 \end{bmatrix} \succeq 0 \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le x \le 1, \\ x \ge y^2, \\ x - x^3 - y^2 \ge 0 \end{array} \right\}$$



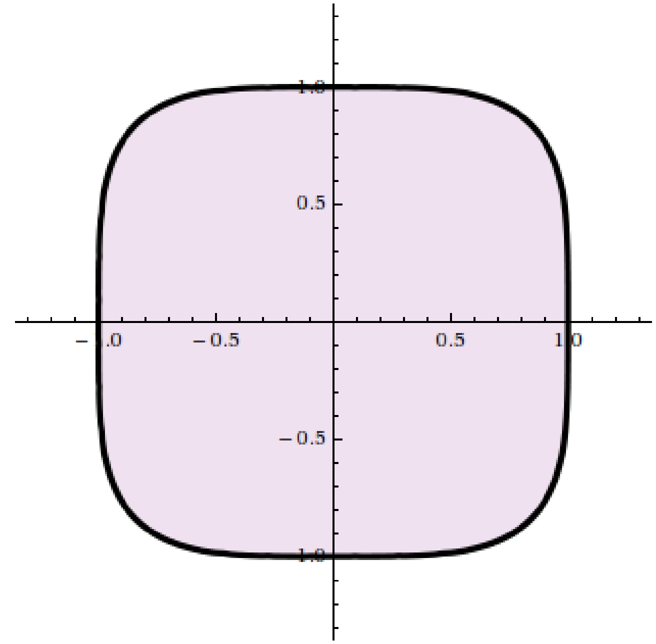
Open Problem: Can every convex semi-algebraic set be written as a spectrahedron or a projection of one?

Exact conditions known in $\mathbb{R}^2$ (Helton-Vinnikov)

no obstructions known when $n > 2$.

TV screen : $I = \langle x^4 + y^4 - 1 \rangle$

$\text{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is not a spectrahedron but is the projection of one. (Helton-Vinnikov)



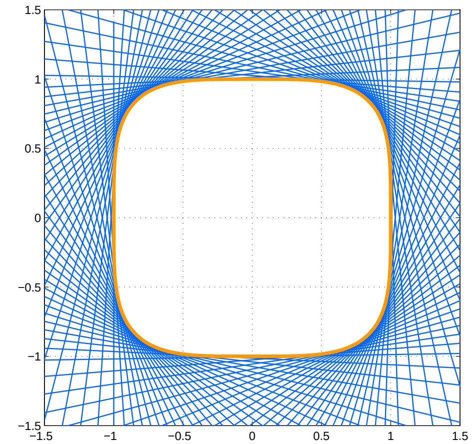
In many more cases $\text{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is the projection of a spectrahedron:

- $\mathcal{V}_{\mathbb{R}}(I)$ finite (Parrilo, Lasserre, Laurent)
- $\mathcal{V}_{\mathbb{R}}(I)$ compact with certain smoothness & curvature (Helton-Nie)
- non-compact examples with $\dim(\mathcal{V}_{\mathbb{R}}(I)) \leq 2$ (Scheiderer)
- some ideals generated by convex quadrics (Gouveia, Parrilo, T.)
- rationally parametrized curves & some hypersurfaces (Didier)

III. Approximating $\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))$ via SDP

$\star$ $\operatorname{conv}(\mathcal{V}_{\mathbb{R}}(I))$ cut out by all linear $f \in \mathbb{R}[\mathbf{x}]$, non-negative on $\mathcal{V}_{\mathbb{R}}(I)$

$\star$ $f \equiv \sum h_j^2 \bmod I \Rightarrow f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$; say f is sum of squares (sos) mod I & k-sos if $\deg(h_j) \leq k$



Definitions: $f \in \mathbb{R}[\mathbf{x}]$

- I is k-sos if $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I) \Rightarrow f$ k-sos mod I
- I is $(1,k)$-sos if $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$ & f linear $\Rightarrow f$ k-sos mod I

Lovász: Which ideals are $(1,1)$-sos, $(1,k)$-sos?

☺ (Parrilo) I zero-dim & radical $\Rightarrow I$ is $|\mathcal{V}_{\mathbb{C}}(I)|$-sos $\Rightarrow (1, |\mathcal{V}_{\mathbb{C}}(I)|)$-sos.

☹ $\langle x^2 \rangle$ not $(1,k)$-sos for any k

Theta bodies of polynomial ideals (Gouveia-Parrilo-T)

$$\text{TH}_k(I) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0 \; \forall \; f \text{ linear \& k-sos mod } I\}$$

$$\text{TH}_1(I) \supseteq \text{TH}_2(I) \supseteq \cdots \supseteq \overline{\text{conv}(\mathcal{V}_\mathbb{R}(I))} \text{ —}(*)$$

Theorem (GPT): $\text{TH}_k(I)$ is the projection of a spectrahedron

Definition: I is

- TH_k-exact if $\text{TH}_k(I) = \text{cl}(\text{conv}(\mathcal{V}_\mathbb{R}(I)))$ (finite convergence)
- TH-exact if there is convergence in $(*)$

$*$ $\langle x^2y - 1\rangle$ is $(1,2)$-sos and TH_2-exact

$*$ $\langle x^2\rangle$ is TH_1-exact

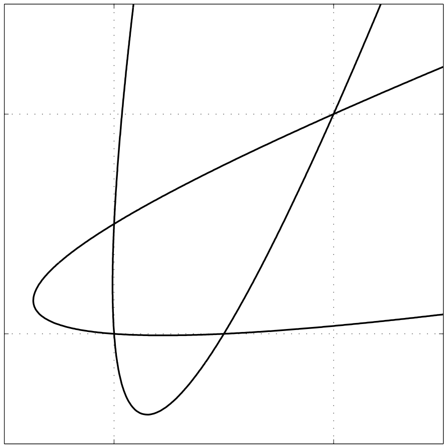
$*$ $\mathcal{V}_\mathbb{R}(I)$ compact $\Rightarrow$ I is TH-exact (Schmüdgen) eg. tv screen

$*$ ideal of rational normal curve of degree d is TH_d-exact

Geometry of theta bodies

Theorem (GPT): $\mathrm{TH}_1(I) = \bigcap\{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(F)) : F \text{ convex quadric in } I\}$

Ex. $I =$ Vanishing ideal of $\{(0,0),(1,0),(0,1),(2,2)\}$



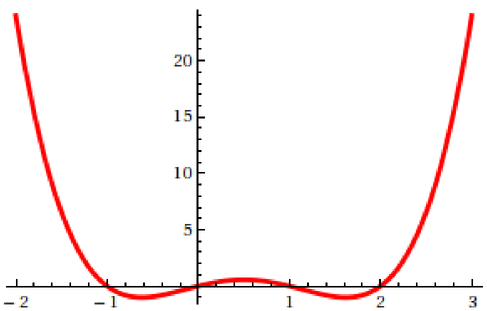
Corollary: $J_n := \langle \sum_{i=1}^n x_i^2 - 1 \rangle$ is $(1,1)$-sos & TH_1-exact.

By Scheiderer $\exists\ f \geq 0$ mod J_n, $n \geq 4$ that is not sos mod J_n.

Question: What is the geometry of higher theta bodies?

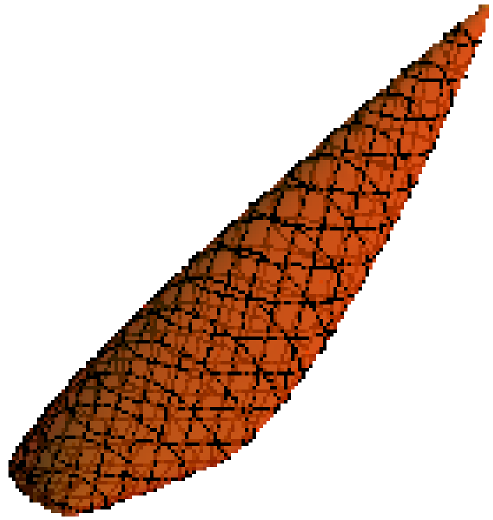
The spectrahedra upstairs

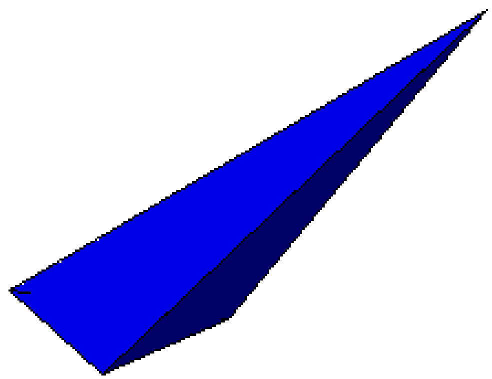
$y_2 \geq y_1^2$



$x^4 - 2x^3 - x^2 + 2x = 0$

roots: $-1, 0, 1, 2$





- $y_2 \geq y_1^2$
- $y_2 y_4 \geq y_3^2$
- $y_4 \geq y_2^2$,
- $2y_3y_2 + y_2^2$
  $-2y_2y_1 - y_3^2$
  $-2y_3y_1^2 - y_2y_1^2$
  $+2y_1^3 + 2y_3y_2y_1$
  $-y_2^3 \geq 0$

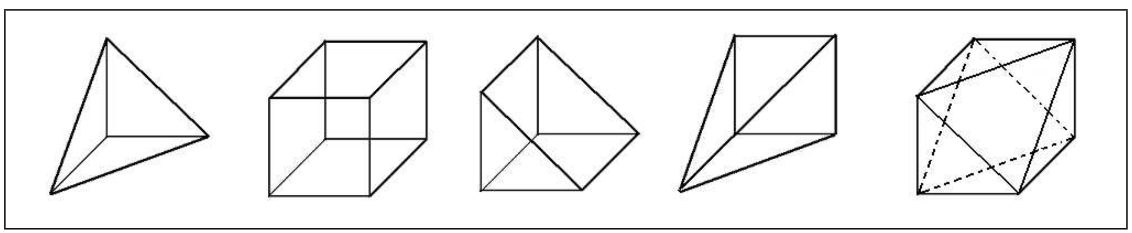
tetrahedron

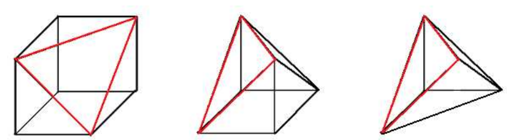
conv(roots lifted to $(1, x, x^2, x^3)$)

Real radical ideals

Definition: I real radical if it is the vanishing ideal of $\mathcal{V}_{\mathbb{R}}(I)$

- Theorem (GPT): I real radical $\Rightarrow$ I is $(1,k)$-sos $\Leftrightarrow$ I is TH$_k$-exact
- Theorem (GPT): I real radical & 0-dimensional. Then TFAE:
  - I is TH$_1$-exact
  - conv$(\mathcal{V}_{\mathbb{R}}(I))$ has a (finite) linear inequality description in which $\forall$ $f(\mathbf{x}) \geq 0$, $f \equiv f^2$ mod I
  - $\cdots \forall$ $f(\mathbf{x}) \geq 0$, $\mathcal{V}_{\mathbb{R}}(I) \subseteq \{f(\mathbf{x}) = 0\} \cup \{f(\mathbf{x}) = 1\}$





IV. Computation

via Combinatorial Moment Matrices (Laurent, Lasserre)

μ probability measure supported on $K \subseteq \mathbb{R}^n$:

- $y_\alpha := \int \mathbf{x}^\alpha d\mu$ moment of order α
- $y := (y_\alpha : \alpha \in \mathbb{N}^n)$ moment sequence of μ
- $M(\mathbf{y}) \in \mathbb{R}^{\mathbb{N}^n \times \mathbb{N}^n}$: $M(\mathbf{y})_{(\alpha,\beta)} := y_{\alpha+\beta}$ moment matrix $(M(\mathbf{y}) \succeq 0)$

$\mathcal{B}$ linear basis for $\mathbb{R}[\mathbf{x}]/I$ containing $1, x_1, \ldots, x_n$ (use Gröbner bases)

- Combinatorial moment matrix: $\mathbf{y} = (y_\gamma : \mathbf{x}^\gamma \in \mathcal{B}) \longrightarrow M_{\mathcal{B}}(\mathbf{y})$

$\mathbf{x}^\alpha \mathbf{x}^\beta \equiv \sum_{\mathbf{x}^\gamma \in \mathcal{B}} \lambda_\gamma \mathbf{x}^\gamma$; $M_{\mathcal{B}}(\mathbf{y})_{\alpha,\beta} := \sum_{\mathbf{x}^\gamma \in \mathcal{B}} \lambda_\gamma y_\gamma$

Theorem (GPT): $\mathrm{TH}_k(I)$ is the closure of the projn onto $(y_1, \ldots, y_n)$ of

$$\left\{ \mathbf{y} \in \mathbb{R}^{\mathcal{B}_{2k}} : M_{\mathcal{B}_k}(\mathbf{y}) \succeq 0,\ y_0 = 1 \right\} \qquad \text{(sdp)}$$

Duality

$\mathbb{R}[\mathbf{x}]^* \cong \{\mathbf{y} := (y_\alpha \in \mathbb{R} : \alpha \in \mathbb{N}^n)\}$ dual vector space of $\mathbb{R}[\mathbf{x}]$

$$
\begin{array}{ccc}
\mathcal{P} = \{p : p(\mathbf{x}) \geq 0, \ \forall \, \mathbf{x} \in \mathbb{R}^n\} & \supseteq & \Sigma = \{\sum h_j^2 : h_j \in \mathbb{R}[\mathbf{x}]\} \\
\updownarrow \, * & & \updownarrow \, * \\
\mathcal{M} := \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}^n} : \mathbf{y} \text{ mom seq in } \mathbb{R}^n\} & \subseteq & \mathcal{M}_{\succeq} := \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}^n} : M(\mathbf{y}) \succeq 0\}
\end{array}
$$

Haviland 1935 $\mathcal{P} = \mathcal{M}^*$, $\mathcal{P}^* = \mathcal{M}$

Berg,Christensen,Jensen 1979: $\Sigma = \mathcal{M}_{\succeq}^*$, $\Sigma^* = \mathcal{M}_{\succeq}$

- $(n = 1) \Rightarrow$ $\mathcal{M} = \mathcal{M}_{\succeq}$ Hamburger's theorem; $\mathcal{P} = \Sigma$ pre Hilbert

V. Why does Lovász care?

- $G = ([n], E)$ graph,
- $S \subseteq [n]$ stable set in G if $\forall i, j \in S, \{i, j\} \notin E$

max stable set problem: $\max\{|S| : S \text{ stable in } G\}$

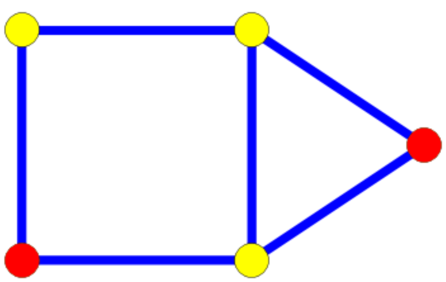
geometric approach: $\text{STAB}(G) := \text{conv}\{\chi^S : S \text{ stable set in } G\}$

max stable set problem: $\max\{\sum x_i : \mathbf{x} \in \text{STAB}(G)\}$ (NP-hard)

$$I_G := \langle x_i - x_i^2 : i \in [n],\ x_i x_j : ij \in E \rangle \Rightarrow \mathcal{V}(I_G) = \{\chi^S : S \text{ stable in } G\}$$

Lovász 1980: introduced $\text{TH}_1(I_G)$ (Lovász theta body of G)

- $\text{STAB}(G) = \text{TH}_1(I_G) \Leftrightarrow G$ perfect
- poly time algorithm when G perfect
- initiated sdp relaxations in combinatorial opt



Max Cut Problem

$C \subseteq E$ cut in G if $\exists\, V_1 \cup V_2 = [n] : \forall ij \in E,\ i \in V_1$ & $j \in V_2$ or vice-versa

max cut problem: $\max\{|C| : C \text{ cut in } G\}$

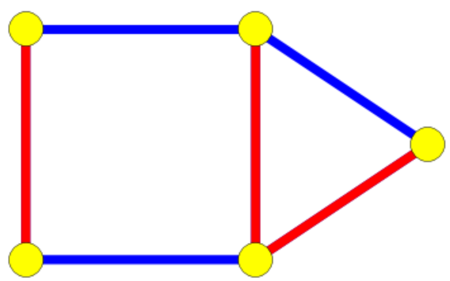
$$\chi^C : (\chi^E)_e = \begin{cases} -1 & \text{if } e \in C \\ 1 & \text{if } e \notin C \end{cases}$$

$\text{CUT}(G) := \text{conv}\{\chi^C : C \text{ cut in } G\}$

max cut problem: $\max\{\sum \frac{1}{2}(1 - x_{ij}) : \mathbf{x} \in \text{CUT}(G)\}$ (NP-hard)

$IG := I(\{\chi^C : C \text{ cut in } G\})$

GB: $\{x_{ij}^2 - 1\} \cup \{\mathbf{x}^A - \mathbf{x}^B : A \cup B \text{ circuit in } G\}$



Theorem (GPT+Laurent): G is cut-perfect (i.e., $\text{CUT}(G) = \text{TH}_1(IG)$) $\Leftrightarrow$

G has no K_5-minor and chordless circuits of length ≥ 5. (a Lovász qn)

Bigger context in which CAG lives

Real algebraic geometry: study of semi-algebraic sets in $\mathbb{R}^n$, preorders in $\mathbb{R}[\mathbf{x}]$, non-negative polynomials, sos polynomials, positivstellensatz

Analysis: moment problems, functional analysis

Optimization: semidefinite programming, polynomial optimization, control theory, combinatorial optimization

FRG project (2008-11):

Semidefinite Optimization and Convex Algebraic Geometry

Helton, Nie, Parrilo, Sturmfels, T., Rostalski, Klep, Gouveia, Vinzant, Diwedi ...