From the Stable Set Problem to Convex Algebraic Geometry

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October '08

Outline

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- If all weights are one, we're searching for $\alpha(G)$, the cardinality of the largest independent set;
- this problem is NP-hard in general.

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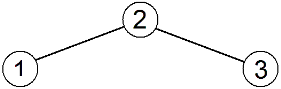
- let $S_G \subset \{0, 1\}^n$ be the collection of all those vectors;
- the polytope $\mathrm{STAB}(G)$ is then defined as the convex hull of the vectors in S_G.

Example



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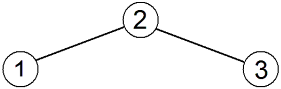

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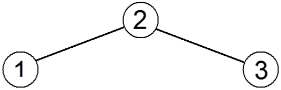

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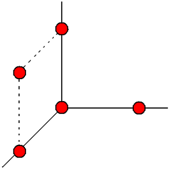

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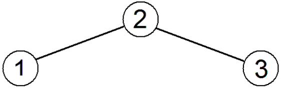

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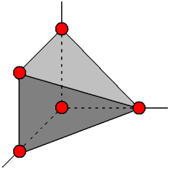

Reformulation of the Problem

Stable Set Problem Reformulated

Given a graph $G = (\{1, ..., n\}, E)$ and a weight vector $\omega \in \mathbb{R}^n$, solve the linear program

$$\alpha(G, \omega) := \max_{x \in \mathrm{STAB}(G)} \langle \omega, x \rangle .$$

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However, finding $\mathrm{STAB}(G)$ is as hard as solving the original problem, and not practical in general.

We intend to find approximations for it.

Fractional Stable Set Polytope

The most common linear relaxation of the stable set polytope is the **fractional stable set polytope** of G, $\mathrm{FRAC}(G)$, to be the set defined by the following inequalities.

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It is possible to optimize over this polytope in polynomial time.

It is in general not a very good relaxation.

Definition of Theta Body

Definition (Lovász ∼ 1980)

Given a graph $G = (\{1, ..., n\}, E)$ we define its theta body, $\mathrm{TH}(G)$, as the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all $(i, j) \in E$.

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- $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)$ since for all stable sets S,

$$0 \preceq (1, \chi_S) \cdot (1, \chi_S)^t = \begin{bmatrix} 1 & \chi_S^t \\ \chi_S & \chi_S \cdot \chi_S^t \end{bmatrix}.$$

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Theorem (Lovász $\sim$ 1980)

The relaxation is tight, i.e. $TH(G) = STAB(G)$, if and only if the graph G is perfect.

Connection to Algebra

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be a polynomial ideal. We call a polynomial ***k*-sos modulo the ideal** I if and only if it can be written as a sum of squares of polynomials of degree at most k modulo I.

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$TH(G) = STAB(G)$ if and only if any linear polynomial $f(\mathbf{x})$ that is non-negative in $STAB(G)$ is 1-sos modulo $\mathcal{I}(S_G)$.

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$TH(G) = STAB(G)$ if and only if any linear polynomial $f(\mathbf{x})$ that is non-negative in $STAB(G)$ is 1-sos modulo $\mathcal{I}(S_G)$.

This property does not depend on the graph, but only on the ideal $\mathcal{I}(S_G)$ and its variety.

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Which ideals are "perfect" i.e., for what ideals I is it true that any linear polynomial that is nonnegative in $\mathcal{V}_{\mathbb{R}}(I)$ is 1-sos modulo I?

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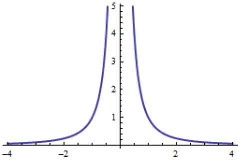
We want to know which ideals are $(1, k)$-sos for some fixed k, and in particular $(1, 1)$-sos.

Example

Consider the ideal $I = \langle yx^2 - 1 \rangle$.



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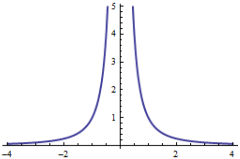

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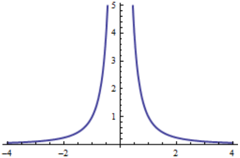


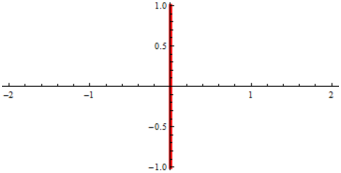
Nonnegative linear polynomials $\longrightarrow y + c^2$ for some real c.

$$y + c^2 \equiv (xy)^2 + (c)^2 \mod I,$$

hence I is $(1,2)$-sos.

Another Example

Consider the ideal $I = \langle x^2 \rangle$.



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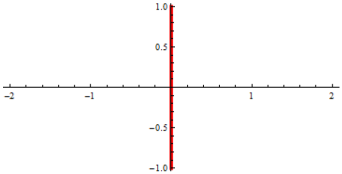

Nonnegative linear polynomials $\longrightarrow \pm x + c^2$ for some real c.

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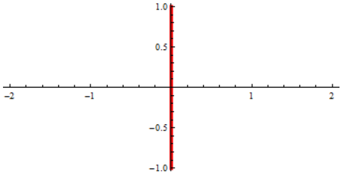

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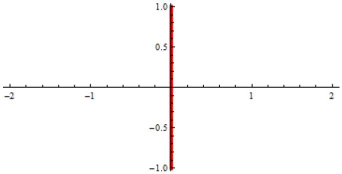

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However x and $-x$ cannot be written as sums of squares hence I is not $(1, k)$-sos for any k.

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A geometric approach to the problem:

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Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define is k-th theta body, $\mathrm{TH}_k(I)$ as the set of all points $\mathbf{p} \in \mathbb{R}^n$ such that for all linear polynomials f that are k-sos modulo I, $f(\mathbf{p}) \geq 0$.

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Remarks:

- $\overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))} \subseteq \cdots \subseteq \mathrm{TH}_k(I) \subseteq \mathrm{TH}_{k-1}(I) \subseteq \cdots \subseteq \mathrm{TH}_1(I)$.

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- $\overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))} \subseteq \cdots \subseteq \mathrm{TH}_k(I) \subseteq \mathrm{TH}_{k-1}(I) \subseteq \cdots \subseteq \mathrm{TH}_1(I)$.
- For any graph G, $\mathrm{TH}_1(\mathcal{I}(S_G)) = \mathrm{TH}(G)$.

Convergence

Recall that a polynomial ideal is **real radical** if and only if $I = \mathcal{I}(\mathcal{V}_{\mathbb{R}}(I))$ i.e., if its real variety is Zariski dense in its complex variety.

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If I is a real radical ideal whose variety is zero-dimensional then $TH_k(I) = \overline{conv(\mathcal{V}_{\mathbb{R}}(I))}$ *for some k.*

Theorem (Scheiderer)

If I is a real radical ideal whose variety is "sufficiently smooth" and one or two dimensional then $TH_k(I) \longrightarrow \overline{conv(\mathcal{V}_{\mathbb{R}}(I))}$.

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We call an ideal **$\mathbf{TH}_k$-exact** if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$.

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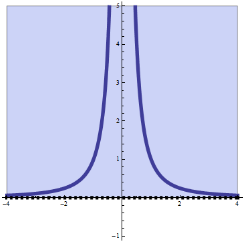
Let I be a real radical ideal. Then I is $(1, k)$-sos if and only if it is TH_k-exact.

The real radical assumption cannot be dropped.
We have seen for $I = \langle x^2 \rangle$ that I is not $(1, k)$-sos, but $\mathrm{TH}_1(I) = \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$.

Theta Bodies and Nonnegativity (continued)

The closure on $\overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$ can also not be dropped.
We have seen for $I = \langle yx^2 - 1 \rangle$ that I is $(1, 2)$-sos but $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ is open.



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Consequences:

- If F is a convex quadric then $\langle F \rangle$ is TH_1-exact.
- There are arbitrarily high dimensional TH_1-exact ideals.

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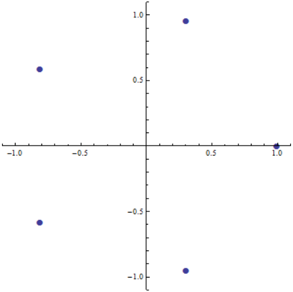

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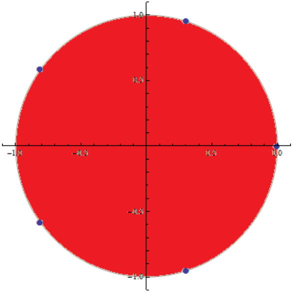

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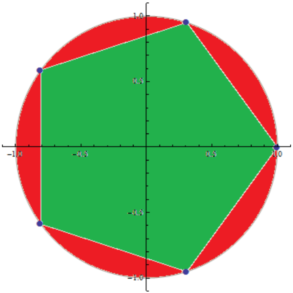

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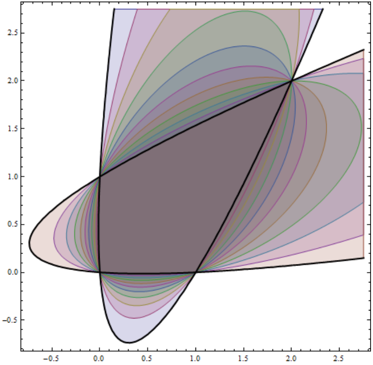

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Theorem

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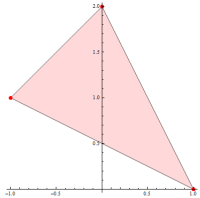
- *I is $(1,1)-sos$;*
- *I is TH_1-exact;*
- *For every facet defining hyperplane H of the polytope $conv(\mathcal{V}_{\mathbb{R}}(I))$ we have a parallel translate H' of H such that $\mathcal{V}_{\mathbb{R}}(I) \subseteq H' \cup H$.*

Examples in $\mathbb{R}^2$



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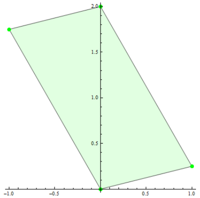
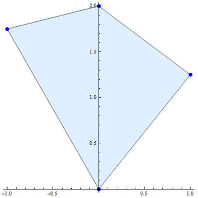
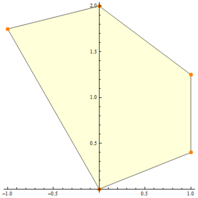


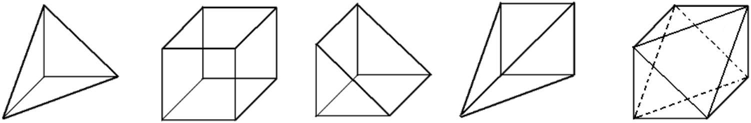


Not TH_1-exact



Examples in $\mathbb{R}^3$

TH_1-exact



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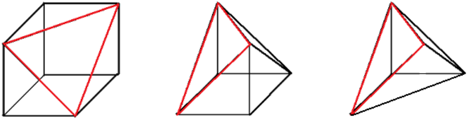

A Small Extension

Theorem

Suppose $S \subseteq \mathbb{R}^n$ is a finite point set such that for each facet F of $conv(S)$ there is an hyperplane H_F such that $H_F \cap conv(S) = F$ and S is contained in at most $t+1$ parallel translates of H_F. Then $\mathcal{I}(S)$ is TH_t-exact.

Consequences

Corollary

Let $S, S' \subset \mathbb{R}^n$ be exact sets (i.e. with TH_1-exact vanishing ideals). Then

- *all points of S are vertices of $conv(S)$,*

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Theorem

If $S \subseteq \mathbb{R}^n$ is a finite exact point set then $\mathrm{conv}(S)$ has at most 2^d facets and vertices, where $d = \dim \mathrm{conv}(S)$. Both bounds are sharp.

Perfect Graphs revisited

Corollary

A graph G is perfect if and only if for any facet supporting hyperplane H of its stable set polytope there is some hyperplane H' parallel to H such that $S_G \subseteq H \cup H'$.

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Corollary

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional down-closed $0/1$-polytope and S be its vertex set. Then S is exact if and only if P is the stable set polytope of a perfect graph.

Combinatorial Moment Matrices I

Let I be a polynomial ideal and

$$\mathcal{B} = \{1 = f_0, f_1, f_2, \ldots\}$$

be a basis of $\mathbb{R}[\mathbf{x}]/I$ and $\mathcal{B}_k = \{f_i : \deg(f_i) \leq k\}$ for all k.

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For all i, j, k define $\lambda_{i,j}^k$ such that

$$f_i f_j \equiv \sum_k \lambda_{i,j}^k f_k.$$

Combinatorial Moment Matrices II

Definition

Given a real vector y indexed by the elements in $\mathcal{B}$, we define the **combinatorial moment matrix** of y as the (possibly infinite) matrix $M_{\mathcal{B}}(y)$ with rows and columns indexed by $\mathcal{B}$ such that

$$[M_{\mathcal{B}}(y)]_{f_i,f_j} = \sum_k \lambda_{i,j}^k y_{f_k}.$$

The k-th **truncated combinatorial moment matrix**, $M_{\mathcal{B}_k}(y)$, is the submatrix of the rows and columns indexed by elements of $\mathcal{B}_k$.

Example

Let $I = \langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \rangle \subset \mathbb{R}[x_1, x_2, x_3]$,

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$$\begin{array}{r} \\ 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1x_2x_3 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \end{array} \\ \left[\begin{array}{cccccccc} y_0 & y_1 & y_2 & y_3 & y_{12} & y_{13} & y_{23} & y_{123} \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & y_{123} & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array}\right] \end{array}$$

Example

$$
\begin{array}{lcccccccccl}
\mathcal{B} = \{ & 1, & x_1, & x_2, & x_3, & x_1x_2, & x_1x_3, & x_2x_3, & x_1x_2x_3 & \} \\
y = (& y_0, & y_1, & y_2, & y_3, & y_{12}, & y_{13}, & y_{23}, & y_{123} &).
\end{array}
$$

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$$
\begin{array}{c}
\\ 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \\ x_1x_2x_3
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\begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_1x_3 & x_2x_3 & x_1x_2x_3 \end{array} \\
\begin{bmatrix}
y_0 & y_1 & y_2 & y_3 & y_{12} & y_{13} & y_{23} & y_{123} \\
y_1 & y_1 & y_{12} & y_{13} & y_{12} & y_{13} & y_{123} & y_{123} \\
y_2 & y_{12} & y_2 & y_{23} & y_{12} & y_{123} & y_{23} & y_{123} \\
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y_{12} & y_{12} & y_{12} & y_{123} & y_{12} & y_{123} & y_{123} & y_{123} \\
y_{13} & y_{13} & y_{123} & y_{13} & y_{123} & y_{13} & y_{123} & y_{123} \\
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Example

$M_{\mathcal{B},1}(y)$ is given by:

	1	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$
1	y_0	y_1	y_2	y_3	y_{12}	y_{13}	y_{23}	y_{123}
x_1	y_1	y_1	y_{12}	y_{13}	y_{12}	y_{13}	y_{123}	y_{123}
x_2	y_2	y_{12}	y_2	y_{23}	y_{12}	y_{123}	y_{23}	y_{123}
x_3	y_3	y_{13}	y_{23}	y_3	y_{123}	y_{13}	y_{23}	y_{123}
x_1x_2	y_{12}	y_{12}	y_{12}	y_{123}	y_{12}	y_{123}	y_{123}	y_{123}
x_1x_3	y_{13}	y_{13}	y_{123}	y_{13}	y_{123}	y_{13}	y_{123}	y_{123}
x_2x_3	y_{23}	y_{123}	y_{23}	y_{23}	y_{123}	y_{123}	y_{23}	y_{123}
$x_1x_2x_3$	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}

Example

$M_{\mathcal{B},2}(y)$ is given by:

	1	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$
1	y_0	y_1	y_2	y_3	y_{12}	y_{13}	y_{23}	y_{123}
x_1	y_1	y_1	y_{12}	y_{13}	y_{12}	y_{13}	y_{123}	y_{123}
x_2	y_2	y_{12}	y_2	y_{23}	y_{12}	y_{123}	y_{23}	y_{123}
x_3	y_3	y_{13}	y_{23}	y_3	y_{123}	y_{13}	y_{23}	y_{123}
x_1x_2	y_{12}	y_{12}	y_{12}	y_{123}	y_{12}	y_{123}	y_{123}	y_{123}
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Theta Bodies and Moment Matrices

Theorem

Let I be a polynomial ideal and choose $\mathcal{B} = \{1, x_1, ..., x_n, ...\}$ as basis for $\mathbb{R}[\mathbf{x}]/I$. Let

$$\mathcal{M}_{\mathcal{B},k}(I) = \{y \in \mathbb{R}^{\mathcal{B}_{2k}} : y_0 = 1; M_{\mathcal{B},k}(y) \succeq 0\}$$

then

$$TH_k(I) = \overline{\pi_{\mathbb{R}^n}(\mathcal{M}_{\mathcal{B},k}(I))}$$

where $\pi_{\mathbb{R}^n} : \mathbb{R}^{\mathcal{B}_{2k}} \to \mathbb{R}^n$ is just the projection over the coordinates indexed by the degree one monomials.

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The closure is really needed as $\pi_{\mathbb{R}^n}(\mathcal{M}_{\mathcal{B},k}(I))$ does not have to be closed.

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Remark:
The closure is really needed as $\pi_{\mathbb{R}^n}(\mathcal{M}_{\mathcal{B},k}(I))$ does not have to be closed. In our example $I = \langle yx^2 - 1\rangle$, we have $\pi_{\mathbb{R}^n}(\mathcal{M}_{\mathcal{B},2}(I))$ to be the open upper half plane, hence not equal to $\mathrm{TH}_2(I)$.

Moment Matrices and Convex Hulls

Theorem (Curto-Fialkow, Laurent)

Given an ideal I and a basis of $\mathbb{R}[\mathbf{x}]/I$

$$\mathcal{B} = \{1 = f_0, x_1 = f_1, x_2 = f_2, ..., x_n = f_n, f_{n+1}, ...\},$$

we can consider the map $\varphi : \mathbb{R}^n \to \mathbb{R}^{\mathcal{B}}$ defined by

$$\varphi_{\mathcal{B}}(p) = (f_0(p), f_1(p), f_2(p), ...),$$

then we have

$$conv\{\varphi_{\mathcal{B}}(p) : p \in \mathcal{V}_{\mathbb{R}}(I)\} = \left\{ y \in \mathbb{R}^{\mathcal{B}} : \begin{array}{l} y_0 = 1, \\ M_{\mathcal{B}}(y) \succeq 0, \\ rk(M_{\mathcal{B}}(y)) < \infty \end{array} \right\}.$$

The Max-Cut Problem

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Given a graph $G = (V, E)$ and a partition V_1, V_2 of V the set C of edges between V_1 and V_2 is called a **cut**.

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Again we will look geometrically at the problem.

The Cut Polytope

Definition

The cut polytope of G, $\mathrm{CUT}(G)$, is the convex hull of the characteristic vectors $\chi_C \subseteq \mathbb{R}^E$ of the cuts of G, where $(\chi_C)_{ij} = -1$ if $(i,j) \in C$ and 1 otherwise.

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Reformulated Problem

Given a vector $\alpha \in \mathbb{R}^E$ solve the optimization problem

$$\mathrm{mcut}(G,\alpha) = max_{x \in \mathrm{CUT}(G)} \frac{1}{2} \langle \alpha, \mathbf{1} - x \rangle .$$

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Computing the vanishing ideal I_G of these characteristic vectors and a basis for its quotient ring, and applying the moment matrix formulation we arrive to a new relaxation for this problem, using theta bodies.

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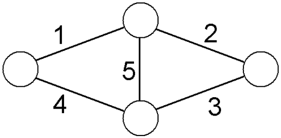
- $U_{e,f} = U_{g,h}$ and $U_{e,g} = U_{f,h}$ if (e, f, g, h) is a 4-cycle;
- The matrix
$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix}$$
is positive semidefinite.

Example



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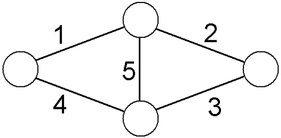

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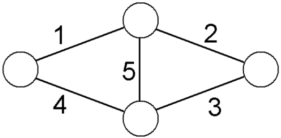
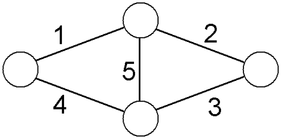


$\mathrm{TH}_1(I_G)$ is the set of $x \in \mathbb{R}^5$ such that there exist y_1 and y_2 such that

	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\succeq 0$

Example



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$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & & & & & \\ x_2 & & & & & \\ x_3 & & & & & \\ x_4 & & & & & \\ x_5 & & & & & \end{bmatrix} \end{array} \succeq 0$$

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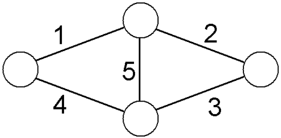

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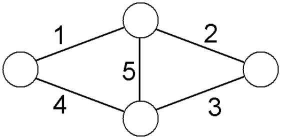

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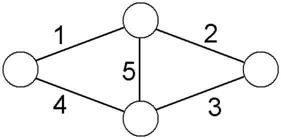

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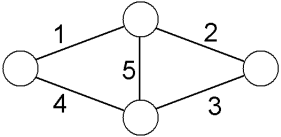

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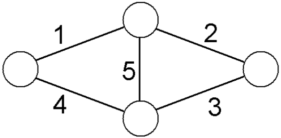

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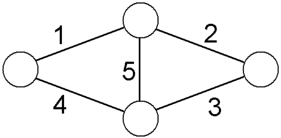

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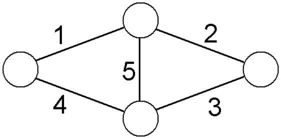

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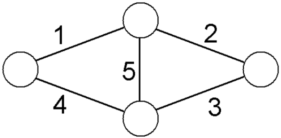

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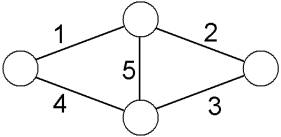

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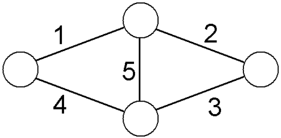

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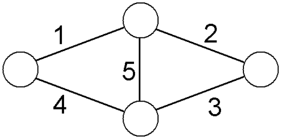

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Using our characterization for TH_1-exact zero-dimensional ideals we get the following characterization, that answers a Lovász question.

Theorem

A graph is cut-perfect if and only if it has no K_5 minor and no chordless cycle of size larger than 4.

Higher Order Theta Bodies

Remarks:

- The higher order theta bodies also have interesting combinatorial descriptions.

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- The higher order theta bodies also have interesting combinatorial descriptions.
- This hierarchy 'refines' a hierarchy obtained by Laurent by a completely different process.

The End

Thank You