1. Let \( \{a_k\} \) and \( \{b_k\} \) be sequences of real numbers. In each of the following, your proof should contain no ellipses [...].

(a) Suppose \( c \in \mathbb{R} \). Use the Distributive Axiom and induction on \( n \) to prove that, for every \( n \in \mathbb{N} \),
\[
\sum_{k=1}^{n} ca_k = c \left( \sum_{k=1}^{n} a_k \right).
\]

(b) Use the Commutative and Associative Axioms and induction on \( n \) to prove that, for every \( n \in \mathbb{N} \),
\[
\sum_{k=1}^{n} (a_k + b_k) = \left( \sum_{k=1}^{n} a_k \right) + \left( \sum_{k=1}^{n} b_k \right).
\]

(WARNING: This seems trivial. It isn’t.)

(c) Suppose \( c \) and \( d \) are real numbers and that \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) both converge. Give a careful proof (using parts (a) and (b), partial sums, and Limit Properties) that \( \sum_{k=1}^{\infty} (ca_k + db_k) \) converges and
\[
\sum_{k=1}^{\infty} (ca_k + db_k) = c \sum_{k=1}^{\infty} a_k + d \sum_{k=1}^{\infty} b_k.
\]

2. In class, we proved that the Geometric Series \( \sum r^k \) converges for \(|r| < 1\) and diverges otherwise. This proof depended on the convergence or divergence of the sequence \( \{r^n\} \). We dealt with the cases in which \(|r| < 1\), \( r = 1 \), and \( r = -1 \). The following problem will take care of the remaining cases: \( r > 1 \) and \( r < -1 \).

**Definition:** Let \( \{a_n\} \) be a sequence of real numbers. We say that \( \lim_{n \to \infty} a_n = \infty \) if, for every \( M > 0 \), there is an \( N \in \mathbb{N} \) such that, if \( n \geq N \), then \( a_n > M \).

(a) Suppose \( \{a_n\} \) is a sequence with \( a_n > 0 \) for all \( n \in \mathbb{N} \). Prove that \( \lim_{n \to \infty} a_n = \infty \) if and only if \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).

(b) We’ve proved that, if \( |c| < 1 \), then \( \lim_{n \to \infty} c^n = 0 \). Prove that, if \( r > 1 \), then \( \lim_{n \to \infty} r^n = \infty \). (Use part (a).)

(c) In each of the following, suppose \( r < -1 \).

i. Prove that \( \{r^n\} \) does not converge to any real number. HINT: Suppose the sequence does converge to a real number and use the following facts to arrive at a contradiction.

   - If \( \{a_n\} \) converges to \( a \), then \( \{|a_n|\} \) converges to \( |a| \). (Proved in class.)
• By definition, if \( \lim_{n \to \infty} a_n = \infty \), then the sequence \( \{a_n\} \) is unbounded and thus \( \{a_n\} \) does not converge to any real number.

ii. Prove that \( \lim_{n \to \infty} r^n \neq \infty \).

iii. Modify the definition above for \( \lim_{n \to \infty} a_n = \infty \) to give a definition for the statement \( \lim_{n \to \infty} a_n = -\infty \) and prove that \( \lim_{n \to \infty} r^n \neq -\infty \).

(d) Summarize the results of this exercise by giving a complete description of the behavior of the sequence \( \{r^n\} \). That is, list the values of \( r \) for which the sequence converges (include the limit of the sequence in your description) and the values of \( r \) for which the sequence diverges.

3. Determine whether each of the following series converges or diverges. If the series converges, give its sum. Give complete and careful proofs of your answers.

(a) \( \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \)

(b) \( \sum_{k=2}^{\infty} \frac{k - 3}{4k + 5} \)

(c) \( \sum_{k=2}^{\infty} \frac{4^k + 1}{5^k} \)

(d) \( \sum_{k=1}^{\infty} a_k \), where \( a_k = \begin{cases} k^2 & \text{if } 1 \leq k \leq 10 \\ 0 & \text{if } k > 10. \end{cases} \)

(Be clear about the sequence of partial sums.)

(e) \( \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \)

HINT: Note that \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \).