Axioms and Elementary Properties of the Field of Real Numbers

When completing your homework, you may use without proof any result on this page, any result we prove in class, and any result you proved in previous homework problems. You must prove any other assertion you wish to use. The axioms are assumed. The propositions may be proved using only the axioms.

- **THE FIELD AXIOMS**

  **Closure under Addition:** If \( a, b \in \mathbb{R} \), then \( a + b \in \mathbb{R} \).

  **Commutativity of Addition:** If \( a, b \in \mathbb{R} \), then \( a + b = b + a \).

  **Associativity of Addition:** If \( a, b, c \in \mathbb{R} \), then \( (a + b) + c = a + (b + c) \).

  **The Additive Identity:** There is a real number, denoted 0, such that \( 0 + a = a + 0 = a \) for all \( a \in \mathbb{R} \).

  **The Additive Inverse:** If \( a \in \mathbb{R} \), then there is a \( b \in \mathbb{R} \) such that \( a + b = 0 \).

  **Closure under Multiplication:** If \( a, b \in \mathbb{R} \), then \( ab \in \mathbb{R} \).

  **Commutativity of Multiplication:** If \( a, b \in \mathbb{R} \), then \( ab = ba \).

  **Associativity of Multiplication:** If \( a, b, c \in \mathbb{R} \), then \( (ab)c = a(bc) \).

  **The Multiplicative Identity:** There is a real number, denoted 1, such that \( 1 \cdot a = a \cdot 1 = a \) for all \( a \in \mathbb{R} \).

  **The Multiplicative Inverse:** If \( a \in \mathbb{R} \) and \( a \neq 0 \), then there is a \( b \in \mathbb{R} \) such that \( ab = 1 \).

  **The Distributive Property:** If \( a, b, c \in \mathbb{R} \), then \( a(b + c) = ab + ac \).

  **The Nontriviality Assumption:** \( 1 \neq 0 \).

  **Substitution of Equals:** If \( a, b, c \in \mathbb{R} \) and \( a = b \), then \( a + c = b + c \) and \( ac = bc \).

- **CONSEQUENCES OF THE FIELD AXIOMS**

  **Proposition A.0:**

  i. The additive identity is unique.

  ii. If \( a \in \mathbb{R} \), then the additive inverse of \( a \) is unique and we denote it \( -a \).

  iii. If \( a, b, c \in \mathbb{R} \) and \( a + c = b + c \), then \( a = b \).

  iv. The multiplicative identity is unique.

  v. If \( a \in \mathbb{R} \) and \( a \neq 0 \), then the multiplicative inverse of \( a \) is unique and we denote it \( a^{-1} \).

  vi. If \( a, b, c \in \mathbb{R} \) and \( c \neq 0 \) and \( ac = bc \), then \( a = b \).

  **Proposition A.1:** If \( a \in \mathbb{R} \), then \( a \cdot 0 = 0 \cdot a = 0 \).

  **Proposition A.2:** If \( a, b \in \mathbb{R} \) and \( ab = 0 \), then \( a = 0 \) or \( b = 0 \).
Proposition A.3: If \(a, b \in \mathbb{R}\), then:

i. \(-(-a) = a\)

ii. \(-a = (-1)a\)

iii. \(-ab = (-a)b\)

iv. \(ab = (-a)(-b)\)

v. \(1 = (-1)(-1)\)

Proposition A.4: If \(a\) and \(b\) are non-zero real numbers, then:

i. \((a^{-1})^{-1} = a\)

ii. \((-a)^{-1} = -a^{-1}\)

iii. \((ab)^{-1} = a^{-1}b^{-1}\)

• THE POSITIVITY AXIOMS AND THEIR CONSEQUENCES

P1: If \(a\) and \(b\) are positive, then \(ab\) and \(a + b\) are also positive.

P2: If \(a \in \mathbb{R}\), then exactly one of the following is true: \(a\) is positive, \(-a\) is positive, or \(a = 0\).

Proposition A.5: If \(a \neq 0\), then \(a^2 > 0\). In particular, \(1 > 0\).

Proposition A.6: If \(a > 0\), then \(a^{-1} > 0\).

Proposition A.7: If \(a > b\) and \(c > 0\), then \(ac > bc\). If \(a > b\) and \(c < 0\), then \(ac < bc\).

Proposition A.8:

i. If \(a, b, c \in \mathbb{R}\) with \(a < b\) and \(b < c\), then \(a < c\).

ii. If \(a, b, c \in \mathbb{R}\) with \(a < b\), then \(a + c < b + c\).

iii. If \(a, b \in \mathbb{R}\) with \(0 < a < b\), then \(0 < b^{-1} < a^{-1}\).

iv. If \(a, b, c, d \in \mathbb{R}\) with \(a < b\) and \(c < d\), then \(a + c < b + d\).

vi. If \(a, b, c, d \in \mathbb{R}\) with \(0 < a < b\) and \(0 < c < d\), then \(ac < bd\).

• THE COMPLETENESS AXIOM

The Completeness of the Real Numbers: Every non-empty subset of \(\mathbb{R}\) that is bounded above has a least upper bound.