1. For each of the following, make a claim about the convergence of the sequence and use the \( \varepsilon-N \) definition of a limit to prove your claim. (Do not apply the Comparison Lemma nor any of the Limit Properties.)

(a) \( \left\{ \frac{4n^2 - 3}{n^2 + 2} \right\} \)

(b) \( \left\{ \left( 1 + \frac{1}{n} \right)^3 \right\} \)

(c) \( \left\{ \sqrt{n^2 + n - n} \right\} \)

2. Prove the following, known as the Squeeze Theorem or the Sandwich Theorem: Let \( \{a_n\}, \{b_n\}, \) and \( \{c_n\} \) be sequences of real numbers. Suppose that there exists a natural number \( K \) such that \( a_n \leq b_n \leq c_n \) for all \( n \geq K \), \( \{a_n\} \) and \( \{c_n\} \) both converge, and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = \ell \). Then \( \lim_{n \to \infty} b_n = \ell \).

3. Suppose \( \{a_n\} \) is a sequence that converges to a real number \( a \). Prove that, if \( a < b \), then there exists an \( N \in \mathbb{N} \) such that, if \( n \geq N \), then \( a_n < b \). (Note that, by symmetric argument, it is also true that, if \( a > b \), then there exists an \( N \in \mathbb{N} \) such that, if \( n \geq N \), then \( a_n > b \).)

4. Prove the Power Property for Limits: Suppose \( \{a_n\} \) converges to \( a \) and \( m \in \mathbb{N} \). Use induction on \( m \) and the Product Property to prove that \( \{a_n^m\} \) converges to \( \{a^m\} \). That is, prove that for all \( m \in \mathbb{N} \),

\[
\lim_{n \to \infty} a_n^m = \left( \lim_{n \to \infty} a_n \right)^m
\]

provided \( \lim_{n \to \infty} a_n \) exists.

5. Prove the Root Property for Limits: Suppose \( \{a_n\} \) converges to \( a \), \( a_n \geq 0 \) for each \( n \in \mathbb{N} \), and \( m \in \mathbb{N} \). Prove that \( \{a_n^{1/m}\} \) converges to \( a^{1/m} \). That is, prove that, if \( \{a_n\} \) is a sequence of non-negative terms, then for all \( m \in \mathbb{N} \),

\[
\lim_{n \to \infty} \sqrt[m]{a_n} = \sqrt[m]{\lim_{n \to \infty} a_n}
\]

provided \( \lim_{n \to \infty} a_n \) exists.

(HINTS:

- First, argue that, under the hypotheses, the number \( a \) must be non-negative.
- Prove the result first in the case that \( a = 0 \).
- To prove the result if \( a > 0 \), use the Difference of Powers Formula:

\[
x^m - y^m = (x - y) \sum_{k=0}^{m-1} x^{m-1-k} y^k
\]

with \( x = a_n^{1/m} \) and \( y = a^{1/m} \) and the Comparison Lemma.

You may use standard properties of exponents like \( (x^\alpha)^\beta = x^{\alpha \beta} \).
6. (a) Let \( a_n = n^{1/n} - 1 \). Note that, for each \( n \in \mathbb{N} \), \( a_n \) is positive and \( n = (a_n + 1)^n \).

Use the Binomial Formula to prove that, for every \( n \in \mathbb{N} \),

\[
n \geq \frac{n(n-1)}{2} a_n^2
\]

and thus \( a_n \leq \sqrt{\frac{2}{n-1}} \) for \( n > 1 \).

(b) Use part (a) to prove that \( \lim n^{1/n} = 1 \).